Complex Numbers (FP1)

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1 Introduction

The numbers you're familiar with are either positive, negative, or zero. We call these the *real* numbers. When you multiply a positive number by a positive number, the resultant number is also positive. When you multiply a negative number by a negative number, the resultant number is again positive. Anything multiplied by zero is zero.

Consequently there is no real number, when squared, that gives a negative number. We can't find a solution to the equation $x = \sqrt{-1}$ if x is a real number. Having a number that satisfies this equation would be useful, both as a purely mathematical curiosity and in almost every field of applied maths. Fortunately we do have such a number, it's called an imaginary number.

The *imaginary unit* is defined by mathematicians to be the solution of $x = \sqrt{-1}$, and is not a real number to avoid contradiction. We denote this solution by i, such that $i^2 = -1$. An *imaginary number* is a multiple of the imaginary unit, in the form bi where b is a real number. With imaginary numbers you can find a square root to all negative numbers, as negative numbers are real multiples of -1. For example:
$\sqrt{-36} = \sqrt{36} \times \sqrt{-1} = 6 \times i = 6i.$

Calling these solutions "imaginary" is just a naming convention and does not detract its legitimacy, we could have called it anything. We defined negative numbers so solve equations such as $x = 2-3$ even though x in this case is not a positive number. How can you have "-1 apples"?! A similar thing with the existance irrational numbers. Numbers are mathematical objects that follow axioms and do have applications in the real world, but they don't have to. Have a look at this SMBC comic (`http://www.smbc-comics.com/index.php?db=comics&id=2013#comic`) to better summarise my half-rant.

A *complex number* is a number in the form $a+bi$ where a and b are real numbers and i is the imaginary unit. Complex numbers cannot be reduced to a single term, the real and imaginary parts remain separate. Mathematicians tend to abbreviate complex numbers by "z", and use subscripts (such as z_1 or z_2) when multiple complex numbers are used. Real numbers and imaginary numbers are both complex, as a real number is just $a + 0i$ and an imaginary number $0 + bi$. When I want to talk about complex numbers in the form $a + bi$ where both a and b are not 0, I will refer to them as *non-trivial complex numbers* (an A-level textbook or exam paper usually means this when they say "complex number"). Here's a Venn diagram to make these annoying distinctions less annoying:

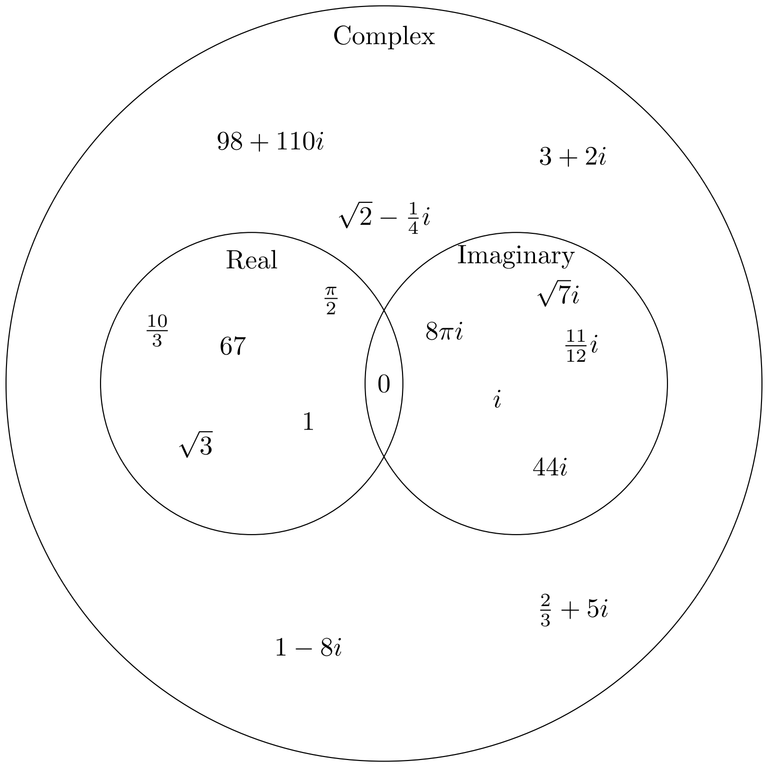

Figure 1: Venn diagram showing that all real and imaginary numbers are complex. Note that 0 is the only number that's both real and imaginary. The numbers outside the Real and Imaginary circles are the non-trivial complex numbers.

2 Operations of complex numbers

We can add, subtract, multiply and divide with complex numbers as well as real numbers, but the rules need some clarification.

2.1 Addition and subtraction

To add two complex numbers together, you just add together the two real parts and two imaginary parts, keeping them separate:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$

Similarly with subtraction:

$$(a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)i$$

This is boring, so let us move on.

2.2 Multiplication

Remember expanding brackets in the early years of secondary school?
"$(x+a)(x+b) = x^2 + (a+b)x + ab$" and all that stuff.
Complex numbers are similar as they involve two terms in brackets, and you can't mix real and imaginary numbers in the same way you can't mix x and x^2.
So here's what happens when you multiply two together:
$(a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + (a_2 b_1 + a_1 b_2)i + b_1 b_2 i^2$
But I just went on about $i^2 = -1$, so we can simplify this:

$$\begin{aligned}(a_1 + b_1 i)(a_2 + b_2 i) &= a_1 a_2 + (a_2 b_1 + a_1 b_2)i + b_1 b_2 i^2 \\ &= a_1 a_2 + (a_2 b_1 + a_1 b_2)i + b_1 b_2(-1) \\ &= a_1 a_2 + (a_2 b_1 + a_1 b_2)i - b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + (a_2 b_1 + a_1 b_2)i\end{aligned}$$

And there's our general formula for complex number multiplication. Note that you always end up with another complex number, where in this case a is $(a_1 a_2 - b_1 b_2)$ and b is $(a_2 b_1 + a_1 b_2)$. You never end up with anything funny because all powers of i simplify to either a real or imaginary number. Here's a table to illustrate the point:

i^0	i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8	i^9	$\cdots$
1	i	-1	$-i$	1	i	-1	$-i$	1	i	$\cdots$

Note the cycle of 1, i, -1, $-i$. This will repeat indefinitely.

2.3 Divison

We need to define another term before we get started: for a complex number $z = a + bi$, its *complex conjugate* is $a - bi$, and is usually denoted by "z^*". For example, if $z = 3 + 2i$, $z^* = 3 - 2i$. Note that:

1. When you add a complex number with its complex conjugate, you always end up with a real number as the imaginary parts cancel:

$$z + z^* = a + bi + a - bi = 2a$$

2. When you multiply a complex number by its complex conjugate, you always end up with a real number from the idea of the difference of two squares:

$$zz^* = (a + bi)(a - bi) = a^2 + abi - abi - b^2 i^2 = a^2 + b^2$$

3. The complex conjugate of a real number is itself, and of an imaginary number its negative (obvious).

With that out of the way, here's the trick to division, best explained first by a worked example. Say we want to divide $8 + 9i$ by $1 + 2i$. We could just write $\frac{8+9i}{1+2i} = \frac{8}{1+2i} + \frac{9}{1+2i} i$ but that isn't in the form $a + bi$ where a and b are real, so we need to do more fiddling with the fraction to get a real denominator and hence in the form that's acceptable. That's where the complex conjugate is useful:

$$
\begin{aligned}
(8+9i) \div (1+2i) &= \frac{8+9i}{1+2i} && (1)\\
&= (\frac{8+9i}{1+2i})(\frac{1-2i}{1-2i}) && (2)\\
&= \frac{8-16i+9i-18i^2}{1+2i-2i-4i^2} && (3)\\
&= \frac{24-7i}{5} && (4)\\
&= \frac{24}{5} - \frac{7}{5}i && (5)
\end{aligned}
$$

Now the explanation: (1) is just writing it as a fraction to make everything neater. In (2) the fraction is being multiplied by the complex conjugate of the denominator divided by itself (tongue twister), which is 1 (and multiplying by 1 does not change the value of the fraction). We multiply the two numerators and two denominators together in (3). As the numerators are two different complex numbers, we end up with another complex number for the new numerator in (4), and as we're multiplying the demoninator by its complex conjugate, we get a real number for the new demoninator in (4) (as discussed in 2. above). As we have a complex number divided by a real number, we can divide the real and imaginary parts by this number separately. And voila, we have our answer in (5).

My explanation may be a little convuluted, but the general method is to multiply the fraction you're trying to evaluate by the complex conjugate of the denominator over itself, and this is analagous to rationalising surds, instead this time we're making the denominator real from complex instead of rational from irrational.

3 Argand diagrams

You're most likely aware of the idea of a number line introduced in primary school:

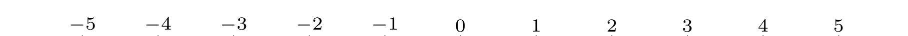

Figure 2: A segment of a number line. We can't draw the whole thing as the whole line continues to ∞ in the positive direction and $-\infty$ in the negative direction

Any real number can be shown as a point on this line, even awkward irrational numbers like $\sqrt{2}$ (somewhere between 1 and 2) and π (somewhere between 3 and 4) have their place, but we don't have room for i or any other non-trivial complex number. Mathematicians get by this using a different diagram akin to number lines that can represent **any** complex number.

An *Argand diagram* (also called a *complex plane*) is a geometrical representation of complex numbers via a Cartesian coordinate system (as in the familiar graphs with an x and y axis). In Argand diagrams the x-axis is the real axis and acts no differently to a normal number line such as the one above. The y-axis is the imaginary axis, and is a number line for the imaginary numbers (except it's vertical). As a complex number has a real part and an imaginary part, it can be represented as a point on an Argand diagram in the same way a real number is a point on a number line. For example:

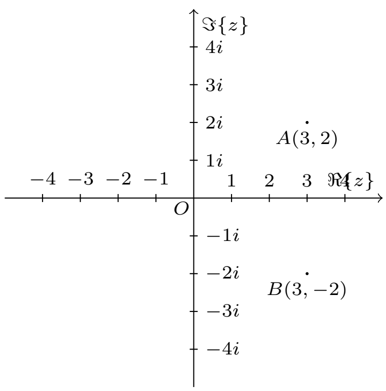

Figure 3: Point A represents $3+2i$ and point B represents its complex conjugate, $3 - 2i$. Note that point B is a reflection of A with the real axis as the line of symmetry, this is the case for all complex conjugates (and is pretty obvious if you think about it).

It's more useful (and you'll find out why) to represent these points instead as vectors from the origin to that point, and I will do so from now on. One reason is that adding complex numbers can be shown easily on an Argand diagram by adding the vectors by the familiar triangle or parallelogram rule. For example, let's add $1 + 2i$ and $3 + i$ together:

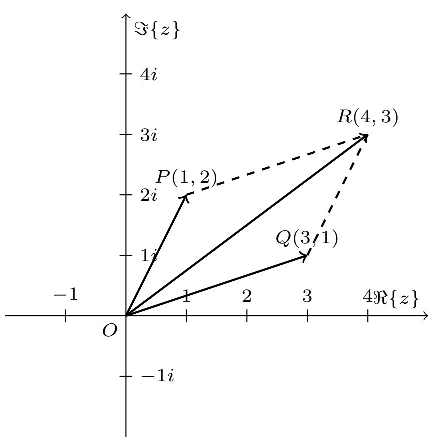

Figure 4: $1+2i$ and $3+i$ are represented by the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, respectively. $\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$ by the triangle rule, giving $4+3i$, the same answer when the numbers are added normally.

3.1 Moduli and arguments

Another reason that makes displaying complex numbers as vectors desirable is that it is easier to make sense of the idea of a *modulus* and an *argument*.

The *modulus* of a complex number is its "absolute value". If a complex number $z = a + bi$ is shown as a vector $\overrightarrow{OP}$ on an Argand diagram then the modulus is the length of $\overrightarrow{OP}$ (denoted "$|\overrightarrow{OP}|$"). Working out the length of vector $\overrightarrow{OP}$ is easy, let $\overrightarrow{OP}$ be the hypotenuse of a right handed triangle with the real part (length a) and the imaginary part (length b) the corresponding sides. By Pythagoras' theorem we'll have $|\overrightarrow{OP}| = \sqrt{a^2 + b^2}$, so therefore the modulus of z, denoted by $|z|$, is equal to $\sqrt{a^2 + b^2}$. $|z|$ is **not** $\sqrt{a^2 + b^2 i^2} = \sqrt{a^2 - b^2}$ because the length of that side is b, not bi (we're dealing with this geometrically and something with an imaginary length doesn't make sense).

The *argument* of a complex number is the angle between the positive real axis and the vector representing the complex number. It can be measured in degrees, but by now you should be familiar with radians as an alternative measure of angles. The conversion is $180\,^\circ = \pi$ radians (usually just π). I will do the next example in degrees to avoid another layer of possible confusion but then on I will be using radian measure so learn to love them and make sure your calculator is set to radians (too many people have mucked up exams from having the wrong angle measure set in their calculators). For a complex number, z, we refer to the argument as $arg(z)$.

A visual example is probably needed to get things straight:

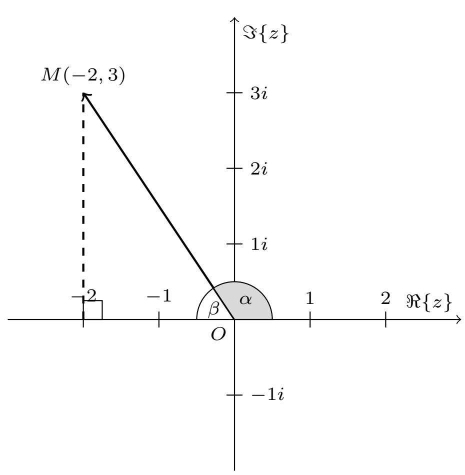

Figure 5: This is the Argand diagram for $z = -2 + 3i$, represented as the vector $\overrightarrow{OM}$. I've added the dashed line and the right angle to make a right-angled triangle out of it.

Modulus:

$|z| = |\overrightarrow{OM}| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$, where $|z|$ is the modulus, and we leave it in surd form.

Agument:

By our definition of an argument, $arg(z)$ is our grey-shaded α, though we first work out β with trigonometry and our right-angled triangle. $\beta = \arctan\frac{3}{2} = 56.3\,^\circ$. From looking at the diagram alone it's obvious that $\alpha + \beta = 180\,^\circ$, so $arg(z) = \alpha = 180\,^\circ - 56.3\,^\circ = 123.7\,^\circ$.

You get a gold star if you noticed my intentional sloppiness. I called the real part "-2" when working out the modulus and "2" when working out the argument. So do signs matter? Yes and no...

For the modulus we are squaring the terms, so signs don't matter as $x^2 = (-x)^2$ for a real number x. 2 may be more "accurate" than -2 as, in the same way imaginary lengths of a triangle make no sense, neither do negative lengths. But do as you please.

The argument is a different story. If I used -2 instead of 2 I would have got $\arctan \frac{3}{-2} = -56.3\,^\circ$ instead of $56.3\,^\circ$. You get the same answer if you add this from $180\,^\circ$ instead of subtract it from $180\,^\circ$. My advice would be to use whichever you want, and work out the agument through common sense (you know α in this case is between $90\,^\circ$ and $180\,^\circ$ from looking at the diagram so you would fiddle with the two numbers to get something between those two values).

Just to clear something up with arguments. I measured α above anticlockwise from the positive axis, but I didn't have to. Let's say we have a complex number with argument γ was $\frac{1}{4}\pi$ rad. If we measured clockwise we could have equally said that γ was $-\frac{7}{4}\pi$ rad. If we measured anticlockwise but went around the whole way and back up again we would have got $2\pi + \frac{1}{4}\pi = \frac{9}{4}\pi$. In general terms our argument for γ can be in the form $2n\pi + \frac{1}{4}\pi$ where n is an integer. I chose $n = 0$ for α because our convention for measuring arguments is that it lies between $-\pi$ and π, and we call this the **principle argument**. An alternative convention would be to measure between 0 and 2π, but follow whatever your course dictates.

3.2 The modulus-argument form

I've been going on about how complex numbers are represented in the form $z = a + bi$ with a and b real numbers, but we have another way of representing the same complex number, called the **modulus-argument form**,

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ which is the modulus and θ the argument of z, and make sure not to forget the i. A simple example can show why it works:

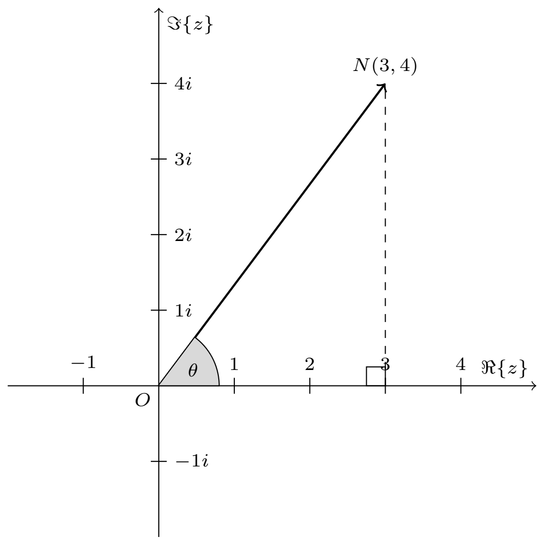

Figure 6: Here the complex number $3+4i$ is represented by the vector $\overrightarrow{OM}$. The modulus, r is $\sqrt{3^2+4^2} = \sqrt{25} = 5$ and the argument $\arctan\frac{4}{3} = 0.93$ rad (save exact values into your calculator)

As the modulus, r is equal to the length of the hypotenuse (by definition), you should be able to work out from the right-angled triangle by trigonometry that the adjacent side (parallel to the real axis) is $r\sin\theta$ and the opposite side (parallel to the imaginary axis) is $r\cos\theta$ and if we plug in the values for r and θ, we should get a and b as they are the lengths of the adjacent and opposite sides, respectively:

$$5(\cos 0.93 + i\sin 0.93) = 5(\frac{3}{5} + \frac{4}{5}i) = 3 + 4i$$

Success!

Just to finish off this section with a handy rule derived from the modulus-argument form that should not be taken for granted (mathematicians have to prove any new information, remember). Say we have two complex numbers z_1 and z_2 with moduli $|z_1| = r_1$ and $|z_2| = r_2$ and arguments θ_1 and θ_2, respectively, therefore:

$$|z_1 z_2| = |z_1||z_2|$$

Proof: $z_1 = r_1(\cos\theta_1 i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 i\sin\theta_2)$ by modulus-argument form, hence:

$$\begin{aligned}
z_1 z_2 &= r_1(\cos\theta_1 i\sin\theta_1) \times r_2(\cos\theta_2 i\sin\theta_2)\\
&= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\\
&= r_1 r_2([\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2] + i[cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2])\\
&= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))
\end{aligned}$$

$z_1 z_2$ is now itself in modulus-argument form and its modulus must be $r_1 r_2$, i.e. $|z_1 z_2|$. You probably don't need to know the proof but it's here to stay. The fourth step comes from the third step by the use of trigonometric identities.

4 Complex numbers as solutions

For a quadratic equation $ax^2 + bx + c = 0$, where a, b and c are real constants, you probably remember the following:

1. If $b^2 - 4ac > 0$ then there are two (real) unique solutions to the equation for x.
2. If $b^2 - 4ac = 0$ then there is one (real) solution to the equation for x (called a repeated root as the solution shows up twice on the graph for the equation).
3. If $b^2 - 4ac < 0$ then there are no (real) solutions to the equation for x.

These rules can easily be derived by substituting these cases into the quadratic formula, which to remind you is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where (1.) gives us $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$, (2.) gives us $x = \frac{-b}{2a}$ and (3.) breaks down because we can't get a real number from a negative square root.

So this is where complex numbers come in. With case (3.), when we plug in a, b and c, the $\frac{-b}{2a}$ part is real (call it p to make it less messy) and the $\frac{\sqrt{b^2-4ac}}{2a}$ is a square root of something less than zero, and so imaginary (call it qi), and we end up with **two** complex numbers because we still have the "$\pm$"bit between them. So the solutions will be $x = p + qi$ and $x = p - qi$, and what do you notice about them? Theyre complex conjugates of each other! This leads to a very important point:

If $x = p + qi$ is a complex solution (root) to a quadratic equation, the other root of the equation will always be its complex conjugate, $p - qi$.

Just to drive the point home, remember how when you factorise a quadratic equation to the form $(x - \alpha)(x - \beta) = 0$, $x = \alpha$ and $x = \beta$ are the solutions to the equation? Okay, not let us multiply out the brackets:

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$$

So in this quadratic equation we have $b = -(\alpha + \beta)$ and $c = \alpha\beta$ (a isn't important right now). I mentioned at the start that b and c are **real** constants, so if our solutions α and β are complex we need these two complex solutions to work in a way that makes the sum of them (b) and the product of them (c) real. I mentioned at the start of Section 2 that two complex conjugates have that exact property. I should mention now that no other non-trivial complex numbers have that property. Therefore we have the imporant bold result above (if you didn't follow this paragraph then don't fuss, as long as you understand the conclusion).