MA244 Analysis III

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April 27, 2013

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Introduction

These lecture notes are a projection of the MA244 Analysis III course 2012/2013, delivered by Dr Claude Baesens at the University of Warwick. The up-to-date version of these notes should be found here:

`http://www.tomred.org/lecture-notes.html`

Students taking this course should also take a look at Alex Wendland's Dropbox notes:

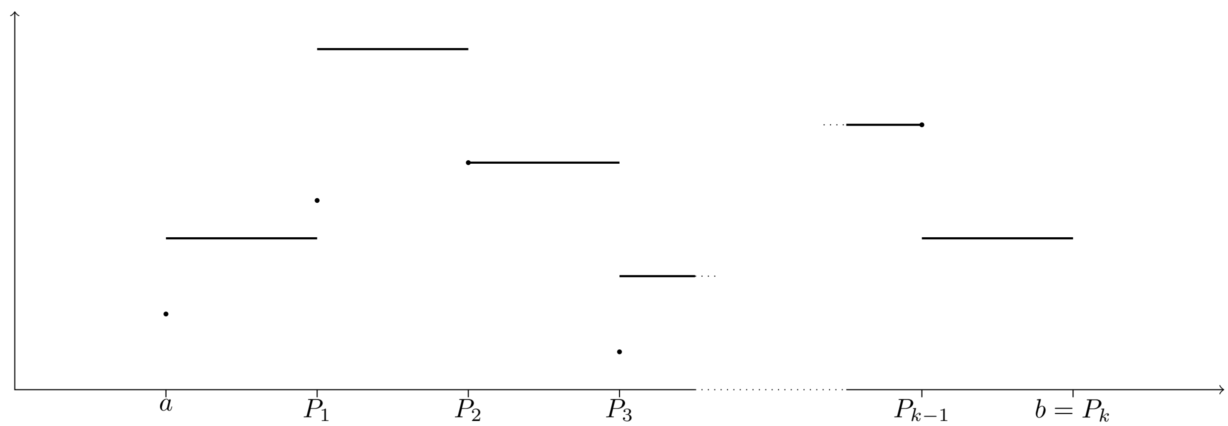
`https://www.dropbox.com/sh/5m63moxv6csy8tn/iRnmC5Vfi_/Year%202/Analysis%20III`

These notes are, to my knowledge, complete, but the tedious treasure hunt of errors will always be an open game. If you spot an error, or you want the source code to fiddle with the notes in your way, e-mail me at `me@tomred.org`. Writing these up has been a benefit to me (there aren't many foolproof ways to avoid proper work), but most of all I hope they're helpful, and good luck!

Tom ♡

1 The Integral for Step Functions

Definition 1: For $a \leq b$ in $\mathbb{R}$, the function φ: $[a,b] \to \mathbb{R}$ is called a *step function* if there is a finite set of points $P \subset (a,b)$, called a *partition*, so that φ is constant on each subinterval of $(a,b) \setminus P$.



If the points of P are numbered in order such that $a =: P_0 < P_1 < P_2 < ... < P_{k-1} < P_k := b$, where $k = |P| - 1$, then $[a,b]$ is partitioned into k subintervals $[P_{j-1}, P_j]$, $1 \leq j \leq k$.

We define C_j be the constant value of $\varphi|_{(P_{j-1},P_j)}$. If P, Q are partitions and $P \subset Q$, we say that Q is a *refinement* of P. Also the *common refinement* of two partitions, P and Q, is $P \cup Q$.

Proposition 1: Fix $a \leq b \in \mathbb{R}$. The set of functions $f : [a,b] \to \mathbb{R}$ is a real vector space, say W. The subset $\mathcal{B}[a,b]$ of bounded functions, the subset $\mathcal{C}[a,b]$ of continuous functions and the subset $\mathcal{S}[a,b]$ of step functions are vector subspaces of W with $\mathcal{S}[a,b] \subset \mathcal{B}[a,b]$, and $\mathcal{C}[a,b] \subset \mathcal{B}[a,b]$.

Proof.

- Let $f, g : [a,b] \to \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}$. Then $\lambda f + \mu g : [a,b] \to \mathbb{R}$ is defined $\forall x \in [a,b]$ by $(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$. As this is defined, addition of f and g and scalar multiplication with these functions is defined. This is the definition of a vector space, so $f : [a,b] \to \mathbb{R}$ is a real vector space, W.

- $$\begin{aligned} f, g \in \mathcal{B}[a,b] \to \mathbb{R} &\implies \forall x \in [a,b],\ |f(x)| \leq K \text{ and } |g(x)| \leq L \text{ for some } K, L \geq 0 \\ &\implies \forall x \in [a,b],\ |(\lambda f + \mu g)(x)| = |\lambda f(x) + \mu g(x)| \\ &\qquad \leq |\lambda||f(x)| + |\mu||g(x)| \\ &\qquad \leq |\lambda|K + |\mu|L \end{aligned}$$
  $\therefore \lambda f + \mu g \in \mathcal{B}[a,b]$, and also $f \equiv 0 \in \mathcal{B}[a,b]$
  $\therefore \mathcal{B}[a,b]$ is a vector subspace of W.

- If φ and ψ are step functions constant on the open intervals of partitions P, Q respectively then φ, ψ and so $\lambda\varphi, \mu\psi$ are constant on the open intervals of $P \cup Q$ so $\lambda\varphi + \mu\psi \in \mathcal{S}[a,b]$ so $\mathcal{S}[a,b]$ is a vector subspace of W. If $\varphi \in \mathcal{S}[a,b]$ it takes at most $2k+1$ values, i.e. $\{C_j : i \leq j \leq k\} \cup \{\varphi(P_j) : 0 \leq j \leq k\}$ so $\varphi \in \mathcal{B}[a,b]$.
  $\therefore \mathcal{S}[a,b]$ is a vector subspace of W.
  $\therefore \mathcal{S}[a,b] \subset \mathcal{B}[a,b]$

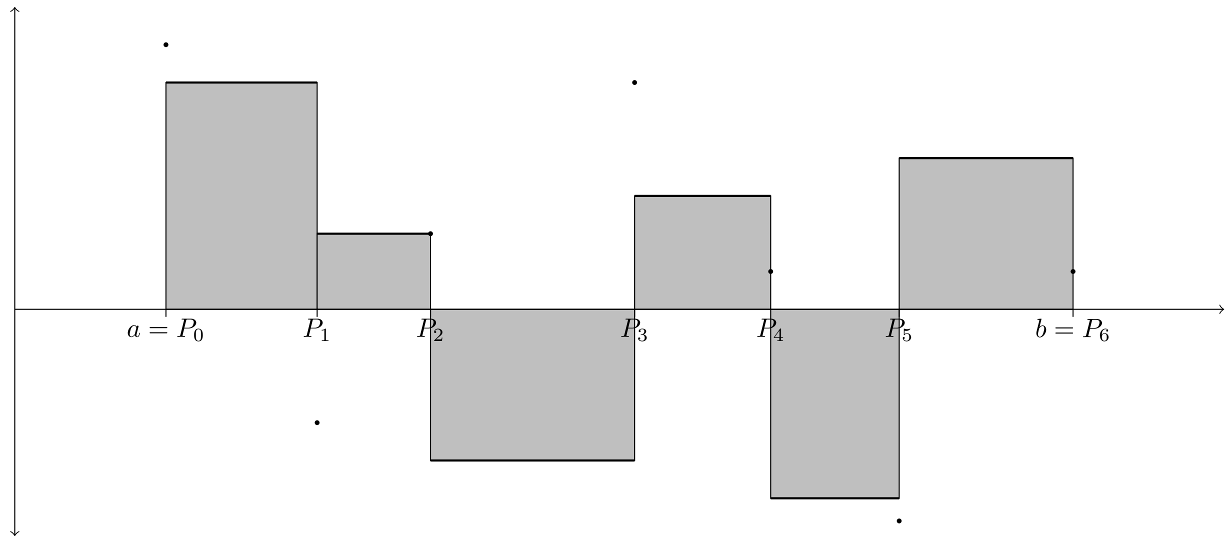
- Recall from Analysis II that $f, g : [a,b] \to \mathbb{R}$ cts $\implies \lambda f + \mu g$ cts. Also cts $f : [a,b] \to \mathbb{R}$ is bounded. Thus $\mathcal{C}[a,b] \subset \mathcal{B}[a,b]$. □

Definition 2: Let $\varphi : [a,b] \to \mathbb{R}$ be a step function constant on the open intervals of a partition $P = \{P_1, P_2, ...P_{k-1}\}$ with $\forall j \in \{1,...,k\} \forall x \in (P_{j-1}, P_j)), \varphi(x) = C_j$. Define

$$\int_a^b \varphi := \sum_{j=1}^{k} C_j (P_j - P_{j-1})$$

Note: This definition ignores $\varphi(P_j)$ and if $C_j < 0$ then the contribution is negative.

For example, take this step function on interval $[a,b]$ with six partitions, with the shading corresponding to the area.



Lemma 2: For a step function $\varphi : [a,b] \to \mathbb{R}$, $\int_a^b \varphi$ is independent of the partition.

Proof. If P and Q are partitions for which φ is constant on their open intervals it suffices to show that the integral of φ is the same using P and $R := P \cup Q$ (because then also the same using Q and R). When comparing P and R, it suffices to add the points one at a time. Thus consider P and $P \cup \{r\}$ where $P_{i-1} < r < P_i$.
$P_{i-1} < x < r \implies \varphi(x) = C_i$, $r < x < P_i \implies \varphi(x) = C_i$, and $C_i(r - P_{j-1}) + C_i(P_i - r) = C_i(P_i - P_{i-1})$. The other summands $\sum_{j \neq 1} C_j(P_j - P_{j-1})$ are the same for P and $P \cup \{r\}$ so $\int_a^b \varphi$ is the same using P and $P \cup \{r\}$. □

Now we establish properties of this integral of step functions, additivity, linearity, bounds and the Fundamental Theorem of Calculus.

Let $\varphi : [a,b] \to \mathbb{R}$ be a step function and $a \leq u < v < w \leq b$. Then $\varphi|_{[u,w]}$ is a step function using $P \cap (u,w)$.

Proposition 3 (additivity): $\varphi \in \mathcal{S}[a,b]$ satisfies $\int_u^w \varphi = \int_u^v \varphi + \int_v^w \varphi$.

Proof. Take partition $u = P_0 < P_1 < ... < P_{k-1} < P_k = v = q_0 < q_1 < ... < q_i = w$ so that φ is constant on each open interval, C_j on (P_{j-1}, P_j) and C'_j on (q_{j-1}, q_j). By Lemma 2:

$$\int_u^w \varphi = \sum_{j=1}^{k} C_j(P_j - P_{j-1}) + \sum_{j-1}^{i} C'_j(q_j - q_{j-1}) = \int_u^v \varphi + \int_v^w \varphi$$

Note that this gives $\int_a^b \varphi = \int_{P_0}^{P_1} \varphi + \int_{P_1}^{P_2} \varphi + ... + \int_{P_{k-1}}^{P_k} \varphi$ where each $\varphi|_{(P_{j-1}, P_j)}$ is a constant function so Proposition 3 reduces integrating step functions to integrating constant functions. Convention: $\int_u^u \varphi = 0$ then Proposition 3 holds for $u \leq v \leq w$. □

Proposition 4 (linearity): Let $\varphi, \psi \in \mathcal{S}[a,b]$ and $\lambda, \mu \in \mathbb{R}$. Then:

$$\int_a^b (\lambda\varphi + \mu\psi) = \lambda \int_a^b \varphi + \mu \int_a^b \psi \tag{$*$}$$

Hence $I : \mathcal{S}[a,b] \to \mathbb{R}$, $I(\varphi) := \int_a^b \varphi$ is a linear map.

Proof. As in Lemma 2 let $R = P \cup Q$ a partition so that on each open interval (r_{j-1}, r_j), φ, ψ and so $\lambda\varphi + \mu\psi$ are constant. Proposition 3 reduces the proof of $(*)$ to its proof on each such interval where it is obvious. Hence I is linear. □

Proposition 5 (fundamental theorem of calculus for step functions): Let $\varphi \in \mathcal{S}[a,b]$ with $\varphi|_{(P_{j-1},P_j)} = C_j$, $1 \leq j \leq k$ where $a = p_0 < p_1 < ... < p_{k-1} < p_k = b$
Then $\Phi : [a,b] \to \mathbb{R}$, $\Phi(x) := \int_a^x \varphi$ is differentiable on

$$\bigcup_{j=1}^{k} (P_{j-1}, P_j) \text{ and } \forall x \in \bigcup (P_{j-1}, P_j),\ \Phi'(x) = \varphi(x)$$

Proof. For $1 \leq j \leq k$, $\forall x \in (P_{j-1}, P_j)$, by Proposition 3:

$$\Phi(x) = \int_a^x \varphi = \int_a^{P_{j-1}} \varphi + \int_{Pj-1}^x \varphi = \int_a^{Pj-1} \varphi + C_j(x - P_{j-1}) = \text{const.} + C_j x$$

So $\varphi|_{(P_{j-1},P_j)}$ is differentiable with derivative C_j. □

Example:

$$\varphi(x) = \begin{cases} 2, & \text{if } 0 \leq x \leq 1 \\ -1, & \text{if } 1 < x \leq 3 \end{cases}$$

$$\Phi = \int_0^x \varphi = \begin{cases} 2x, & \text{if } x \leq 0 \leq 1 \\ 2(1-0) + (-1)(x-1) = 3 - x, & \text{if } 1 < x \leq 3 \end{cases}$$

1. Φ is differentiable on $[0,1) \cup (1,3]$ with $\Phi' = \varphi$.
2. Φ is continuous but not differentiable at $x = 1$.
3. Changing $\varphi(1)$ does not affect 1. and 2.

Definition 3: For $f \in \mathcal{B}[a,b]$ write $||f||_\infty := \sup_{x\in[a,b]} |f(x)|$, called the *supremum norm* of f.

Example: In $\mathcal{B}[0,2\pi]$, $||\sin||_\infty = 1$, $||\cos||_\infty = 1$, $||\sin - \cos||_\infty = \sqrt{2}$

Proposition 6 (bounds for the integral): Let $a \leq b$ and $\varphi \in \mathcal{S}[a,b]$ with $\forall x \in [a,b]$, $m \leq \varphi(x) \leq M$. Then:

$$m(b-a) \leq \int_a^b \varphi \leq M(b-a), \text{ furthermore: } \left| \int_a^b \varphi \right| \leq ||\varphi||_\infty (b-a)$$

Proof.

- Take $a = P_0 < P_1 < ... < P_{k-1} < P_k = b$ s.t. $\forall j \in \{1,...,k\}$, $\forall \in (P_{j-1}, P_j)$, $\varphi(x) = C_j$. Then $m \leq C_j \leq M$ and $m(P_j - P_{j-1}) \leq C_j(P_j - P_{j-1}) \leq M(P_j, P_{j-1})$. Adding these inequalities gives:

$$m(b-a) \leq \int_a^b \varphi \leq M(b-a)$$

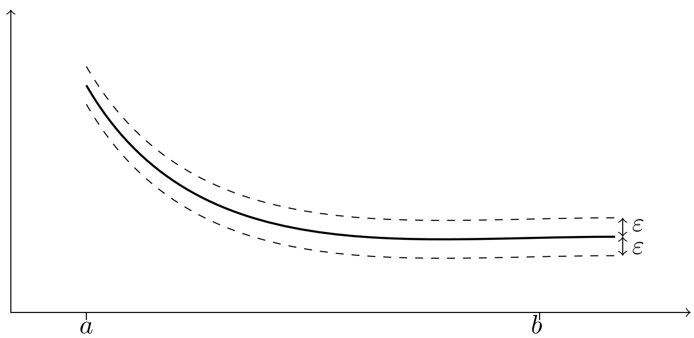
- Since $\forall x \in [a,b]$, $-\|\varphi\|_\infty \leq \varphi(x) \leq \|\varphi\|_\infty$ we have $|\int_a^b \varphi| \leq \|\varphi\|_\infty (b-a)$ □

Note: If $v \leq w$, it is sometimes useful to define $\int_w^v \varphi = -\int_v^w \varphi$ but $|\int_a^b \varphi| \leq \|\varphi\|_\infty (b-a)$ requires $a \leq b$.

2 The Integral for Regulated Functions

Definition 4:

1. A function $f : [a, b] \to \mathbb{R}$ is *regulated* if $\forall \varepsilon > 0$, $\exists \varphi \in \mathcal{S}[a, b]$ s.t. $\|\varphi - f\|_\infty < \varepsilon$
2. Equivaletly, f is regulated if $\exists$ a sequence $(\varphi_n)_{n=1}^\infty$ in $\mathcal{S}[a, b]$ s.t. $\|\varphi_n - f\|_\infty \to 0$ as $n \to \infty$.



Proposition 7: Fix $a \leq b \in \mathbb{R}$. The set of regulated functions $[a, b] \to \mathbb{R}$ forms a vector subspace $\mathcal{R}[a, b]$ of W and $\mathcal{S}[a, b] \subset \mathcal{R}[a, b] \subset \mathcal{B}[a, b]$.

Proof.

- Let $f, g \in \mathcal{R}[a, b]$ and choose sequences $(\varphi_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty$ in $\mathcal{S}[a, b]$ s.t. $\|\varphi_n - f\|_\infty \to 0$ as $n \to \infty$. Then $(\lambda\varphi_n + \mu\psi_n)_{n=1}^\infty$ is a sequence in $\mathcal{S}[a, b]$ by Proposition 1, and

$$\begin{aligned} \|(\lambda\varphi_n + \mu\psi_n) - (\lambda f + \mu g)\|_\infty &= \sup_{x\in[a,b]} |(\lambda\varphi_n + \mu\psi_n)(x) - (\lambda f - \mu g)(x)| \\ &= \sup_{x\in[a,b]} |\lambda(\varphi_n - f)(x) + \mu(\psi_n - g)(x)| \\ &\leq \sup_{x\in[a,b]} \{|\lambda| \cdot |(\varphi_n - f)(x)| + |\mu| \cdot |(\varphi_n - g)(x)|\} \\ &\leq |\lambda| \sup_{x\in[a,b]} |(\varphi_n - f)(x)| + |\mu| \sup_{x\in[a,b]} |(\varphi_n - g)(x)| \\ &\leq |\lambda| \|\varphi_n - f\|_\infty + |\mu| \|\psi_n - g\|_\infty \to 0 \text{ as } n \to \infty \end{aligned}$$

  Thus $\lambda f + \mu g \in \mathcal{R}[a, b]$. Clearly $\mathcal{S}[a, b] \subset \mathcal{R}[a, b]$.

- Given $f \in \mathcal{R}[a, b]$ take $\varphi \in \mathcal{S}[a, b]$ with $\|\varphi - f\|_\infty = 1$, so:

$$-1 \leq \varphi(x) - f(x) \leq 1$$
$$-\|\varphi\|_\infty - 1 \leq \varphi(x) - 1 \leq f(x) \leq \varphi(x) + 1 \leq \|\varphi - f\|_\infty + 1$$

  Hence $f \in \mathcal{B}[a, b]$. □

Proposition 8: If $f : [a, b] \to \mathbb{R}$ is continuous then it is regulated.
(Recall that $f : [a, b] \to \mathbb{R}$ is continuous if it is continuous at each $c \in [a, b]$. It is continuous at c if $\forall \varepsilon > 0$, $\exists \delta_c(\varepsilon)$ s.t. $x \in (c - \delta_c(\varepsilon), c + \delta_c(\varepsilon)) \cap [a, b] \implies f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon)$.

Proof. Choose $\varepsilon > 0$ and put $A = \{t \in [a, b] : \exists \psi_t \in \mathcal{S}[a, t] \text{ with } \|f|_{[a,t]} - \psi_t\|_\infty < \varepsilon\}$. Now $A \supset [a, a + \frac{1}{2}\delta_a(\varepsilon)] \cap [a, b]$ so $A \neq \emptyset$. Also $t \in A \implies [a, t] \subset A$. Let $c := \sup A$. Then $c \in [a, b]$. We want to show that $c = b$. Suppose not, i.e. $a < c < b$. Now $[c - \frac{1}{2}\delta_c(\varepsilon), c + \frac{1}{2}\delta_c(\varepsilon)] \cap [a, b] \subset A$ using $\psi_{c-\frac{\delta_c}{2}}$ for $[a, c - \frac{\delta_c}{2}]$ and $f(c)$ on $[c - \frac{1}{2}\delta_c(\varepsilon), c + \frac{1}{2}\delta_c(\varepsilon)] \cap [a, b]$ contradicting $c = \sup A$.
$\therefore A = [a, b]$ □

Definition 5: For a regulated function $f : [a, b] \to \mathbb{R}$ define $\int_a^b f$ to be:

$$\lim_{n\to\infty} \int_a^b \varphi_n$$

where $(\varphi_n)_{n=1}^\infty$ is a sequence of step functions that converge uniformly to f on $[a, b]$ in the sense $\|\varphi_n - f\|_\infty \to 0$ as $n \to \infty$. This requires:

Proposition 9: Let $f : [a, b] \to \mathbb{R}$ be a regulated function and $(\varphi_n)_{n=1}^\infty$ be a sequence of step functions converging uniformly to f. Then $(\int_a^b \varphi_n)_{n=1}^\infty$ converges in $\mathbb{R}$. If $(\psi_n)_{n=1}^\infty$ is also a sequence in $\mathcal{S}[a, b]$ that converges uniformly to f then $\lim_{n\to\infty} \int_a^b \psi_n = \lim_{n\to\infty} \int_a^b \varphi_n$.

Proof.

- $\forall \varepsilon > 0\ \exists N(\varepsilon)$ s.t. $\forall n \geq N(\varepsilon)\ \|\varphi_n - f\|_\infty < \varepsilon$. If $m, n \geq N(\varepsilon)$ then

$$\begin{aligned}\|\varphi_n - \varphi_m\|_\infty &= \sup_{x\in[a,b]} \{|(\varphi_n(x) - f(x)) + (f(x) - \varphi_m(x))|\} \\ &\leq \sup_{x\in[a,b]} \{|\varphi_n(x) - f(x)| + |f(x) - \varphi_m(x)|\} \\ &\leq \|\varphi_n - f\|_\infty + \|f - \varphi_m\|_\infty \leq 2\varepsilon\end{aligned}$$

  Now:

$$\left|\int_a^b \varphi_n - \int_a^b \varphi_m\right| = \left|\int_a^b (\varphi_n - \varphi_m)\right| \leq (b - a)\|\varphi_n - \varphi_m\|_\infty \qquad \text{(by Proposition 6)}$$

  So $(\int_a^b \varphi_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ and hence converges.

- $\omega_{2k-1} := \varphi_k$, $\omega_{2k} := \psi_k$ $(k = 1, 2, ...)$ gives another sequence of step functions converging uniformly to f so $\left(\int_a^b \omega_n\right)_{n=1}^\infty$ converges and its subsequences $\left(\int_a^b \varphi_n\right)_{n=1}^\infty$ and $\left(\int_a^b \psi_n\right)_{n=1}^\infty$ converge to its limit. □

Let us show that the integral of regulated functions has good properties.

Proposition 10 (additivity): Let $a \leq u \leq v \leq w \leq b$ and let $f : [a, b] \to \mathbb{R}$ be a regulated function. Then $f|_{[u,w]}$ is a regulated function. Moreover:

$$\int_u^w f = \int_u^v f + \int_v^w f$$

Proof. Choose a sequence of step functions $\varphi_n : [a, b] \to \mathbb{R}$ with $\|\varphi_n - f\|_\infty \to 0$ as $n \to \infty$. $\|\varphi|_{[u,w]} - f|_{[u,w]}\|_\infty \leq \|\varphi_n - f\|_\infty$ so $(\varphi_n|_{[u,w]})$ is a sequence of step functions converging uniformly to $f|_{[u,w]}$, which is therefore regulated. The same applies to $[u, v]$ and $[v, w]$. Hence:

$$\begin{aligned}\int_u^v f + \int_v^w f &:= \lim_{n\to\infty} \int_u^v \varphi_n + \lim_{n\to\infty} \int_v^w \varphi_n && \text{(by Proposition 9)} \\ &= \lim_{n\to\infty} \left(\int_u^v \varphi_n + \int_v^w \varphi_n\right) && \text{(by Analysis I)} \\ &= \lim_{n\to\infty} \int_u^w \varphi_n && \text{(by Proposition 3)} \\ &=: \int_u^w f\end{aligned}$$

□

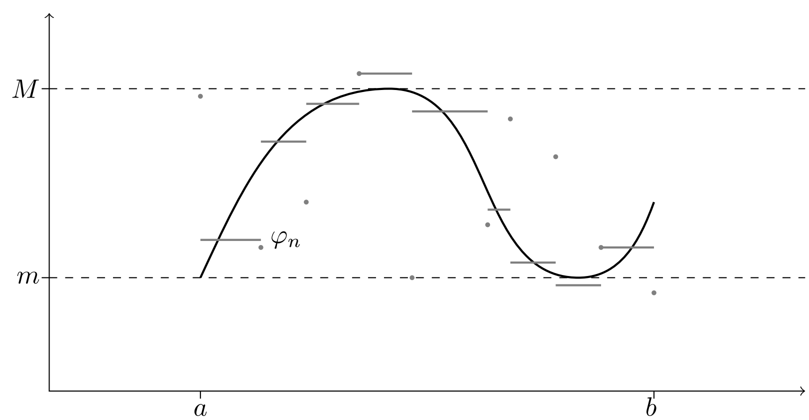
Proposition 11 (linearity): $I : \mathcal{R}[a, b] \to \mathbb{R}$, $I(f) = \int_a^b f$ is linear.

Proof. For $f, g \in \mathcal{R}[a,b]$ take sequence $(\varphi_n)_{n=1}^{\infty}$, $(\psi_n)_{n=1}^{\infty}$ in $\mathcal{S}[a,b]$ converging uniformly to f, g, respectively. Then $(\lambda\varphi_n + \mu\psi_n)_{n=1}^{\infty}$ converges uniformly to $\lambda f + \mu g$ by Proposition 7 and

$$\begin{aligned} I(\lambda f + \mu g) &= \lim_{n\to\infty} I(\lambda\varphi_n + \mu\psi_n) \\ &= \lim_{n\to\infty} (\lambda I(\varphi_n) + \mu I(\psi_n)) && \text{(by Proposition 4)} \\ &= \lambda \lim_{n\to\infty} I(\varphi_n) + \mu \lim_{n\to\infty} (\psi_n) && \text{(by Analysis I)} \\ &= \lambda I(f) + \mu I(g) && \square \end{aligned}$$

Proposition 12 (bounds): If $f \in \mathcal{R}[a,b]$ and $\forall x \in [a,b]$ $m \le f(x) \le M$ then $m(b-a) \le \int_a^b f \le M(b-a)$. Also $|\int_a^b f| \le \|f\|_\infty (b-a)$.

Proof. Let $(\varphi_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{S}[a,b]$ converging uniformly to f.

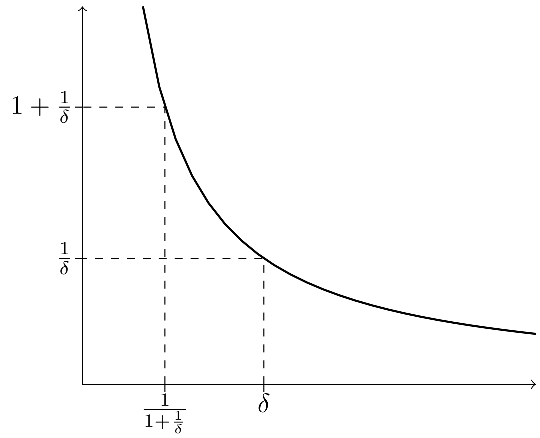


Get new φ_n by replacing any value of φ_n that are greater than M by M and less than m by m. This cannot increase $\|\varphi_n - f\|_\infty$ by Proposition 6 $\forall n \in \mathbb{N}$. $m(b-a) \le \int_a^b \varphi_n \le M(b-a)$. Let $n \to \infty$ give $m(b-a) \le \int_a^b f \le M(b-a)$. Now use $m = -\|f\|_\infty$ and $M = \|f\|_\infty$. $\square$

Definition 6: $f : A \subset \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t. $x, y \in A$, $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. "Uniformly" means consistently on the domain.

Note: Here one δ works for any x, y. This is *not* the same as continuity where $\forall c \in A$, $\forall \varepsilon > 0$, $\exists \delta = \delta_c(\varepsilon) > 0$ s.t. $y \in A$, $|y - c| < \delta_c(A) \implies |f(y) - f(c)| < \varepsilon$
But uniformly continuous $\implies$ continuous (take $\forall c$, $\delta_c(\varepsilon) = \delta(\varepsilon)$).

Example: $f : (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is *not* uniformly continuous because for $\varepsilon < 1$ any δ satisfies $\left|\delta - \frac{1}{1+\frac{1}{\delta}}\right| < \delta$ but $\left|f(\delta) - f\left(\frac{1}{1+\frac{1}{\delta}}\right)\right| = 1 \not< \varepsilon$ so no δ satisfies the required property for uniform continuity.



Theorem 13: Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then f is uniformly continuous.

Proof. Suppose, if possible, that $f : [a, b] \to \mathbb{R}$ is continuous, but not uniformly. Then $\exists \varepsilon_0$ s.t. $\forall \delta > 0$, $\exists x, y \in [a, b]$ which depend on δ s.t. $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$.

For $n = 1, 2, 3, ...$ consider $\delta = \frac{1}{n}$. Then $\exists x_n, y_n \in [a, b]$ s.t. $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. The sequence $(x_n)_{n=1}^{\infty}$ is bounded so by the Bolzano-Weierstrass Theorem has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ say, and $x_{n_k} \to u$, say, as $k \to \infty$. Note that $u \in [a, b]$. Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \to 0$ as $k \to \infty$, we also have $y_{n_k} \to u$ as $k \to \infty$.

Now f is continuous so by equivalent definition of continuity $f(x_{n_k}) \to f(u)$ as $k \to \infty$ and $f(y_{n_k}) \to f(u)$ as $k \to \infty$. Which contradicts $\forall \delta$, $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0 > 0$ □

Question: where does this proof break down in the last example?

Example: $(0, 1) \to \mathbb{R}$, $x \mapsto \sqrt{x}$ is uniformly continuous because it is $g|_{(0,1)}$ where $g : [0, 1] \to \mathbb{R} : g(x) = \sqrt{x}$ is uniformly continuous.

Corollary 14: Let $f : [a, b] \to \mathbb{R}$ be continuous and $\varepsilon > 0$. Then $\exists \varphi \in \mathcal{S}[a, b]$ with $\|\varphi - f\|_\infty \leq \varepsilon$. Thus f is regulated as in Proposition 8.

Proof. By Theorem 13 f is uniformly continuous. Take $\delta = \delta(\varepsilon) > 0$ s.t. $x, y \in [a, b]$, $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.
Take k with $\frac{(b-a)}{k} < \delta$ and put $P_j = a + \frac{(b-a)j}{k}$, $0 \leq j \leq k$. Put $\varphi(P_j) = f(P_j)$ and $\forall x \in (P_{j-1}, P_j)$, $\varphi(x) := f(P_j)$, then $\forall x \in [a, b]$, $\exists j \in \{0, ..., k\}$ s.t. $x \in (P_{j-1}, P_j]$ and then $|\varphi(x) - f(x)| = |f(P_j) - f(x)| < \varepsilon$ because $|P_j - x| < \delta$ so $\|\varphi - f\|_\infty = \sup_{x \in [a,b]} |\varphi(x) - f(x)| \leq \varepsilon$

□

3 The Indefinite Integral and the Fundamental Theorem of Calculus

Definition 7: Let $f : [a,b] \to \mathbb{R}$ be regulated. Define the *indefinite integral* $F : [a,b] \to \mathbb{R}$ by:

$$F(x) := \int_a^x f$$

Note that $F(a) = 0$.

Proposition 15: The indefinite integral $F : [a,b] \to \mathbb{R}$ of $f \in \mathcal{R}[a,b]$, $F(x) := \int_a^x f$, satisfies $\forall x, y \in [a,b]$, $|F(y) - F(x)| \leq \|f\|_\infty |y - x|$ and is uniformly continuous.

Remark: Say F is *Lipschitz* with *Lipschitz constant* L if it increases distance by no more than a constant factor L. Here $L = \|f\|_\infty$.

Note: $[0,1] \to \mathbb{R} : x \mapsto \sqrt{x}$ is not Lipschitz.

Proof.

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_a^y f - \int_a^x f \right| \\ &= \left| \int_x^y f \right| && \text{(by Proposition 12)} \\ &\leq \|f\|_\infty |y - x| \end{aligned}$$

If $f = 0$ then F is constant so is uniformly continuous. Otherwise $\|f\|_\infty > 0$ and given $\varepsilon > 0$ put $\delta > \frac{\varepsilon}{2\|f\|_\infty}$. Then $\forall x, y \in [a,b]$, $|x - y| < \delta \implies |F(x) - F(y)| \leq \frac{\varepsilon}{2} < \varepsilon$, so F is uniformly continuous. □

Theorem 16: Let $f \in \mathcal{R}[a,b]$ and suppose f is continuous at $c \in [a,b]$. Then the indefinite integral $F : [a,b] \to \mathbb{R}$ is differentiable at c and $F'(c) = f(c)$.

Proof. $\forall \varepsilon > 0 \ \exists \delta(\varepsilon)$ s.t. $x \in (c - \delta, c + \delta) \cap [a,b] \implies f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. Thus:

$$\begin{aligned} 0 < h < \delta \implies & (f(c) - \varepsilon)h \\ & \leq \int_c^{c+h} f \\ & = F(c+h) - F(c) \\ & \leq (f(c) + \varepsilon)h && \text{(by Proposition 12)} \\ \implies & \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \\ & \leq \varepsilon \end{aligned}$$

□

Corollary 17 (first form of the FTC): Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Then:

1. There exists a differentiable function $g : [a,b] \to \mathbb{R}$ with $g' = f$ (i.e. g is an antiderivative of f).
2. If $h : [a,b] \to \mathbb{R}$ is differentiable with $h' = f$ then $\exists k \in \mathbb{R}$ s.t. $\forall x \in [a,b]$, $h(x) = \int_a^x f + k$

Proof.

1. $g(x) = \int_a^x f$ will do by Theorem 16.
2. $(h - g)' = 0$ so $h - g$ is a constant funtion (by MVT, see Analysis II). □

Note: g differentiable $\not\Longrightarrow$ g' is continuous.

Example: $g(0) = 0$, $g(x) = x^2 \sin \frac{1}{x}$, $(x \neq 0)$, $g'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ $(x \neq 0)$

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0$$

Theorem 18 (second form of the FTC): Let $f : [a, b] \to \mathbb{R}$ be regulated and suppose there exists a differentiable function $g : [a, b] \to \mathbb{R}$ s.t. $g' = f$. Then $\int_a^b f = g(b) - g(a)$. This means an integral is found in terms of an antiderivative.

Note:

1. If f is continuous this follows from Corollary 17.
2. Putting x for b in Theorem 18 gives $F(x) := \int_a^x f = g(x) - g(a)$ and F is differentiable since we assumed f is. For regulated f (e.g. a step function) F need not be differentiable.

Proof. Fix $\varepsilon > 0$ and choose $\varphi \in \mathcal{S}[a, b]$ with $\|\varphi - f\|_\infty \leq \varepsilon$. Take partition $a = P_0 < P_1 < ... < P_{k-1} < P_k = b$ with $\forall j \in \{1, ..., k\}, \forall x \in (P_{j-1}, P_j)$, $\varphi(x) = c_j$. The MVT for $g|_{[P_{j-1}, P_j]}$ gives $x_j \in (P_{j-1}, P_j)$ with $g(P_j) - g(P_{j-1}) = g'(x_j)(P_j - P_{j-1}) = f(x_j)(P_j - P_{j-1})$
$\|\varphi - f\|_\infty$ gives $c_j - \varepsilon \leq f(x_j) \leq c_j + \varepsilon$
Add $(c_j - \varepsilon)(P_j - P_{j-1}) \leq g(P_j) - g(P_{j-1}) \leq (c_j + \varepsilon)(P_j - P_{j-1})$ for $1 \leq j \leq k$ to get

$$\int_a^b \varphi - \varepsilon(b - a) \tag{$*$}$$

Now

$$\begin{aligned} \left| \int_a^b \varphi - \int_a^b f \right| &= \left| \int_a^b (\varphi - f) \right| \\ &\leq \|\varphi - f\|_\infty (b - a) && \text{(by Proposition 12)} \\ &\leq \varepsilon(b - a) && (**) \end{aligned}$$

So $|g(b) - g(a) - \int_a^b f| = 0$ as required. ∎

Note: If f is given by $f(x) = \cos 2x$, say, then $\int_a^b f$ may be written as $\int_a^b f(t)\,dt$ or $\int_a^b \cos 2s\,ds$.

Example: $\forall n \in \mathbb{N}$, $f_n(x) := x^n$; $g_n(x) := \frac{x^{n+1}}{n+1}$ satisfies $g_n' = f_n$ on $\mathbb{R}$. If $a < b$ in $\mathbb{R}$ then $f_n|_{[a,b]}$ is differentiable so is continuous so is regulated, and Theorem 18 gives $\int_a^b f_n = g_n(b) - g_n(a)$ and one might write $\int_a^b f_n = \int_a^b x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b := \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} = g(b) - g(a)$

Corollary 19 (integration by parts): If $F, G := [a, b] \to \mathbb{R}$ are differentiable and $F' =: f$, $G' =: g$ are regulated then $\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG$. To prove this we need the following proposition:

Proposition 20: $f, g \in \mathcal{R}[a, b] \implies fg \in \mathcal{R}[a, b]$

Proof of Proposition 20. $\forall \varepsilon > 0$ take $\varphi, \psi \in \mathcal{S}[a, b]$ s.t. $\|\varphi - f\|_\infty < \varepsilon_1$, $\|\psi - g\|_\infty < \varepsilon_2$ for some ε_1, ε_2, and $\varphi\psi \in \mathcal{S}[a, b]$ (by question 1.8)

$$\begin{aligned} \|fg - \varphi\psi\|_\infty &\leq \|fg - f\psi\|_\infty + \|f\psi - \varphi\psi\|_\infty \\ &\leq \|f\|_\infty \|g - \psi\|_\infty + \|f - \varphi\|_\infty \|\psi\|_\infty \\ &< \|f\|_\infty \varepsilon_2 + \varepsilon_1(\|\psi\|_\infty) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This works if $\varepsilon_1 = \frac{\varepsilon}{2(\|g\|_\infty+1)}$ and $\varepsilon_2 = \min\{\frac{\varepsilon}{2\|f\|_\infty}, 1\}$ □

Proof of Corollary 19. $H := FG$ is differentiable (by Analysis II) with $H' = fG + Fg$. By Proposition 7 and 20 $H' \in \mathcal{R}[a,b]$. Hence by Theorem 18, $F(b)G(b) - F(a)G(a) = \int_a^b fG + \int_a^b Fg$. □

Corollary 21 (integration by substitution): Suppose $f \in \mathcal{C}[a,b]$, g differentiable with g' continuous and g maps $[c,d]$ into $[a,b]$. Then

$$\int_c^d (f \circ g)g' = \int_{g(c)}^{g(d)} f \qquad (\dagger)$$

or

$$\int_c^d (f \circ g)g' = \int_c^d f(g(t))g'(t)dt$$

Proof. Let $F : [a,b] \to \mathbb{R}$ s.t. $F(x) := \int_a^x f$ be differentiable by Corollary 15 (FTC 1). Let $h : [c,d] \to \mathbb{R}$ s.t. $h := F \circ g$. Then h is differentiable with $h' = (F' \circ g)g'$ (by the chain rule, Analysis II). By Theorem 18, $\int_c^d (F' \circ g)g' = \int_c^d h' = h(d) - h(c) = F(g(d)) - F(g(c))$, and the RHS of $(\dagger) = F(g(d)) - F(g(c))$. □

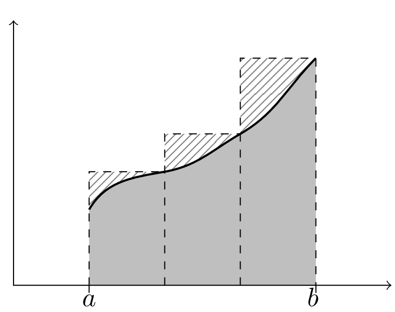
Example: $\int_0^{\frac{\pi}{2}} \cos^3 x \sin x \, dx$

$$f(t) = t^3$$
$$g(t) = \cos t$$
$$g'(t) = -\sin t$$

$\therefore \int_0^{\frac{\pi}{2}} \cos^3 x \sin x \, dx = -\int_{\cos 0}^{\cos\frac{\pi}{2}} u^3 \, du = \int_0^1 u^3 \, du = \frac{1}{4}$

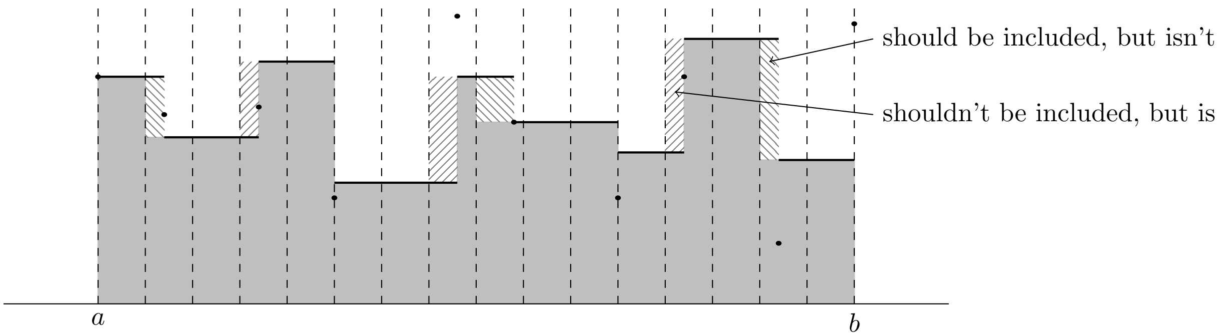
Theorem 22: If $f : [a,b] \to \mathbb{R}$ is regulated, then:

$$\sum_{j=1}^n \frac{b-a}{n} f\left(a + j\frac{b-a}{n}\right) \to \int_a^b f(x)\,dx \text{ as } n \to \infty$$



Proof. We first prove it for step functions, then we extend it to general regulated functions.
Let $\varphi \in \mathcal{S}[a,b]$, φ is constant except for $k-1$ discontinuities. Then:

$$\left|\sum_{j=1}^n \frac{b-a}{n} \varphi\left(a + j\frac{b-a}{n}\right) - \int_a^b \varphi\right| \le (k-1)\frac{b-a}{n}\|\varphi\|_\infty$$



Then:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{b-a}{n}\varphi\left(a+j\frac{b-a}{n}\right)=\int_a^b\varphi,\ \forall\varphi\in\mathcal{S}[a,b]$$

We now extend it to regulated functions.
Let f be regulated, i.e. $\forall\varepsilon>0$, $\exists\varphi\in\mathcal{S}[a,b]$ s.t. $\|f-\varphi\|_\infty\leq\varepsilon$ and $\left|\int f-\int\varphi\right|\leq\varepsilon(b-a)$.

$$\begin{aligned}\left|\sum_{j=1}^{n}\frac{b-a}{n}f\left(a+j\frac{b-a}{n}\right)-\sum_{j=1}^{n}\frac{b-a}{n}\varphi\left(a+j\frac{b-a}{n}\right)\right| &\leq\sum_{j=1}^{n}\frac{b-a}{n}\left|f()-\varphi()\right| \\ &\leq\varepsilon(b-a)\end{aligned}$$

Then:

$$\begin{aligned}\left|\sum_{j=1}^{n}\frac{b-a}{n}f\left(a+j\frac{b-a}{a}\right)-\int_a^b f\right| &\leq\left|\sum_{j=1}^{n}\frac{b-a}{n}f()-\sum_{j=1}^{n}\frac{b-a}{n}\varphi()\right|+\left|\sum_{j=1}^{n}\frac{b-a}{n}\varphi()-\int\varphi\right|+\left|\int\varphi-\int f\right| \\ &\leq 2\varepsilon(b-a)+(k-1)\frac{b-a}{n}\|\varphi\|_\infty\end{aligned}$$

This is small if ε is small, then n large. □

4 (I don't know that this chapter's called)

How do we recognise a regulated function? For instance, are the following functions regulated?

1. $f(x) = |x|$ on $[-1, 1]$
2. $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$
3. $f(x) = \sin \frac{1}{x}$
4. $f(x) = x \sin \frac{1}{x}$

We start by defining the following limits:

1. $f(x-) := \lim_{t \searrow 0} f(x - t)$ if the limit exists.
2. $f(x+) := \lim_{t \searrow 0} f(x + t)$ if the limit exists.

Proposition 23: $f : [a, b] \to \mathbb{R}$ is regulated iff $\forall x \in (a, b)$, $f(x-)$ and $f(x+)$ exist, as well as $f(a+)$ and $f(b-)$.

Proof.

$\Longrightarrow$: Let $f \in \mathcal{R}[a, b]$ and $x \in (a, b)$. We show that $\forall \varepsilon > 0$, we have:

$$\limsup_{t \searrow 0} f(x + t) - \liminf_{t \searrow 0} f(x + t) \leq \varepsilon$$

Then $\limsup = \liminf$ and the limit $t \searrow 0$ of $f(x + t)$ exists.

$\forall \varepsilon > 0$, $\exists \varphi_N$ s.t. $\|f - \varphi_N\|_\infty < \frac{\varepsilon}{2}$
There exists interval (x, δ) where φ_N is constant so that $|f(y) - f(z)| < \varepsilon \ \forall y, z \in (x, \delta)$. Then $\limsup - \liminf \leq \varepsilon$.

$\Longleftarrow$: Adapt the proof of Proposition 8 ($\mathcal{C}[a, b] \subset \mathcal{R}[a, b]$). That is, introduce $\forall \varepsilon > 0$:

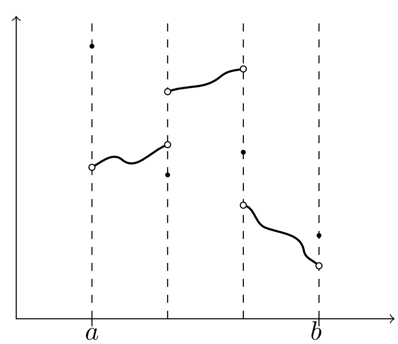
$$A = \{t \in [a, b] : \exists \psi \in \mathcal{S}[a, b] \text{ s.t. } \|f|_{[a,t]} - \psi\|_\infty < \varepsilon\}$$

$A \neq \emptyset$ because $f(a+)$ exists. Let $c = \sup A$. If $c < b$, construct a step function that approximates f beyond c. Then c cannot be smaller than b.

□

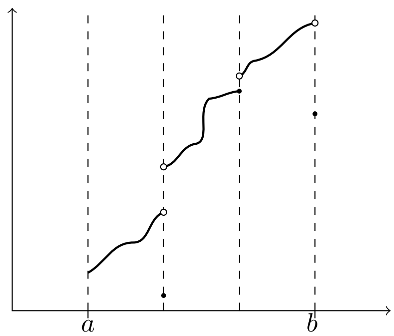
Examples:

1. Piecewise continuous functions: f is piecewise continuous if there exists a partition $a = P_0 < P_1 < P_2 < ... < P_k = b$, such that f is continuous on each interval (P_{j-1}, P_j), and $f(P_j+)$, $f(P_j-)$ exist.



2. Monotone functions: note that f is:

non-decreasing	if	$f(x) \leq f(y)$	$\forall x \leq y$
increasing	if	$f(x) < f(y)$	$\forall x < y$
non-increasing	if	$f(x) \geq f(y)$	$\forall x \leq y$
decreasing	if	$f(x) > f(y)$	$\forall x < y$



3. Devil's Staircase: see chapter 7.

We have defined $\int_a^b f$ for $f \in \mathcal{R}[a,b]$ where $-\infty < a < b < \infty$ and such that f is bounded. Here, we extend the definition to $a = -\infty$ or $b = \infty$, or f unbounded.

Example:

1. $f(x) = \frac{1}{x}$ in $(0,1]$
2. $f(x) = \frac{1}{\sqrt{x}}$ in $(0,1]$
   $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \searrow 0} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \searrow 0} 2x^{\frac{1}{2}}\Big|_\varepsilon^1 = \lim_{\varepsilon \searrow 0} 2 - 2\sqrt{\varepsilon} = 2$
3. $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty} \int_1^N \frac{dx}{x^2} = \lim_{N\to\infty} -\frac{1}{x}\Big|_1^N = \lim_{N\to\infty} -\frac{1}{N} + 1 = 1$

Note:

1. If $f : (a,b] \to \mathbb{R}$ is regulated on $[t,b]$ $\forall t \in (a,b]$, and $\int_t^b f \to L$ as $t \to a+$, then we say that $\int_a^b f$ exists, and it is equal to L.
2. If $f : [a,\infty) \to \mathbb{R}$ is regulated on $[a,N]$ $\forall N > a$ and $\int_a^N f \to L$ as $N \to \infty$, then we say that $\int_a^\infty f$ exists and is equal to L.

Remark: $\int_{-\infty}^\infty f$ exists if both limits $\int_{-\infty}^c f$ and $\int_c^\infty f$ exist for some $c \in \mathbb{R}$.

Let:

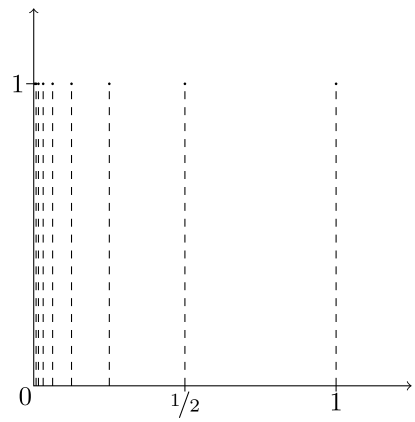
$$U(f) = \inf\left\{\int_a^b \varphi : \varphi \in \mathcal{S}[a,b], \varphi \geq f\right\}$$
$$L(f) = \sup\left\{\int_a^b \varphi : \varphi \in \mathcal{S}[a,b], \varphi \leq f\right\}$$

We say that f is *Riemann integrable* if $U(f) = L(f)$, in which case we define $\int_a^b f = U(f) = L(f)$.

Note: If $f \in \mathcal{R}[a,b]$, then f is Riemann integrable. Why? Take φ s.t. $\|f - \varphi\|_\infty \leq \varepsilon$ so that $\varphi - \varepsilon \leq f \leq \varphi + \varepsilon$ and $\left|\int_a^b (\varphi + \varepsilon) - \int_a^b (\varphi - \varepsilon)\right| \leq 2\varepsilon(b-a)$.

Examples:

1. Non-regulated but Riemann integrable:
   $f(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{4}, \ldots \\ 0 & \text{otherwise} \end{cases}$



   f is not regulated because $f(0+)$ does not exist. f is Riemann integrable with $\int_a^b f = 0$. Clear that $L(f) \geq 0$. $U(f) \leq \int_0^1 \varphi_n$ where $\varphi_n = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{n}] \\ 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{4}, \ldots, \\ 0 & \text{otherwise} \end{cases}$ so then $U(f) \leq \frac{1}{n}$. Then $U(f) \leq \inf_n \frac{1}{n} = 0$.

2. Not Riemann integrable but Lebesque integrable: In $[0, 1]$:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

   ($\int_0^1 f = 0$ for Lebesque)

5 Pointwise Convergence and its Disadvantages

Definition 8: Let $A \subset \mathbb{R}$ and for all $n \geq 1$, let $f_n : A \to \mathbb{R}$. We say that (f_n) *converges pointwise* to f if $\forall x \in A$, $\forall \varepsilon > 0$, there exists $N = N_x(\varepsilon)$ such that $n \geq N_x(\varepsilon) \implies |f_n - f(x)| < \varepsilon$.

Examples:

1. "Odd roots", $f_n : [-1, 1] \to \mathbb{R}$, $f_n(x) = x^{\frac{1}{2n-1}}$.

$$\lim_{n\to\infty} x^{\frac{1}{2n-1}} = 1 \ \forall x > 0$$

$$\text{Let } f(x) = \begin{cases} 1 & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x \in [-1,0) \end{cases}$$

   $\forall$ fixed $x \in [-1, 1]$, we have $\lim_{n\to\infty} f_n(x) = f(x)$

2. "Odd powers", $f_n(x) = x^{2n-1}$ on $[-1, 1]$.

$$\text{Let } f(x) = \begin{cases} 1 & \text{if } x = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_a^b f_n(x) \to \int_a^b f(x)\,dx \ \forall (f_n) \to f \text{ pointwise?}$$

3. $f_n : [0, 1] \to \mathbb{R}$

$$f_n(x) = \begin{cases} 2n^2 x & \text{if } x \in [0, \frac{1}{2n}] \\ 2n - 2n^2 x & \text{if } x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

   $\forall x \in [0, 1] : f_n(x) \to 0$

$$\int_0^1 f_n(x)\,dx = \frac{1}{2n} n = \frac{1}{2} \ \forall n. \text{ Then } \frac{1}{2} \lim_{n\to\infty} \int_0^1 f_n(x) \neq \int_0^1 f(x)\,dx = 0$$

Disadvantages of pointwise convergence:

1. The pointwise limit of a sequence of continuous functions need not be continuous (first and second examples).

2. The integral of a pointwise limit of a sequence of continuous functions need not be the limit of their integrals (example 3).

6 Uniform Convergence: its Advantages for Integrals and Continuity

Definition 9: Let $A \subset \mathbb{R}$. Say a sequence (f_n) of functions $f_n : A \to \mathbb{R}$ *converges uniformly* to the function $f : A \to \mathbb{R}$ if $\forall \varepsilon > 0\ \exists N = N(\varepsilon)$ s.t. $(n \geq N,\ x \in A) \implies |f_n(x) - f(x)| < \varepsilon$. Equivalently, $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, where $\|\cdot\|_\infty$ is the sup norm $\|f_n - f\|_\infty := \sup_{x\in A} |f_n(x) - f(x)|$.
Note: In pointwise convergence, $N_x(\varepsilon)$ can depend on x as well as ε, whereas in uniform convergence, the same $N(\varepsilon)$ must work for every x. Uniform convergence implies pointwise convergence (using $N(\varepsilon)$ for $N_x(\varepsilon)$ for each x). The three examples in the previous chapter show that pointwise convergence does not imply uniform convergence.
Say (f_n) is *uniformly Cauchy* if $\forall \varepsilon > 0$:

$$\exists M = M(\varepsilon) \text{ s.t. } (m, n \geq M(\varepsilon),\ x \in A) \implies |f_n(x) - f_m(x)| < \varepsilon \tag{$*$}$$

In this case $\forall x \in A$, $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ so has a limit in $\mathbb{R}$, call that $f(x)$, which defines $f : A \to \mathbb{R}$. Let $m \to \infty$ in $(*)$ to get $\forall n \geq M(\varepsilon)$, $\forall x \in A$, $|f_n(x) - f(x)| \leq \varepsilon$. So (f_n) converges uniformly to f.
Slogan: "uniform convergence makes the integral converge".

Theorem 24: Suppose $f_n : [a, b] \to \mathbb{R}$ is a sequence of regulated functions and $(f_n) \to f$ uniformly as $n \to \infty$. Then $f : [a, b] \to \mathbb{R}$ is regulated and $\left(\int_a^b f_n\right) \to \int_a^b f$ as $n \to \infty$.
Idea: Find f_N within $\frac{\varepsilon}{2}$ of f and $\varphi \in \mathcal{S}[a, b]$ within $\frac{\varepsilon}{2}$ of f_N.

Proof. Given $\varepsilon > 0$, choose $N = N(\frac{\varepsilon}{2})$ s.t. $(n \geq N,\ x \in [a, b]) \implies |f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$ $(**)$
f_N is regulated so choose $\varphi \in \mathcal{S}[a, b]$ s.t. $x \in [a, b] \implies |\varphi(x) - f_N(x)| \leq \frac{\varepsilon}{2}$. Then $x \in [a, b] \implies |\varphi(x) - f(x)| \leq \varepsilon$. Hence f is regulated. By $(**)$ and Proposition 11 and 12:

$$n \geq N \implies \left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \frac{\varepsilon}{2}(b - a)$$

Hence $(\int_a^b f_n) \to \int_a^b f$ as $n \to \infty$. □

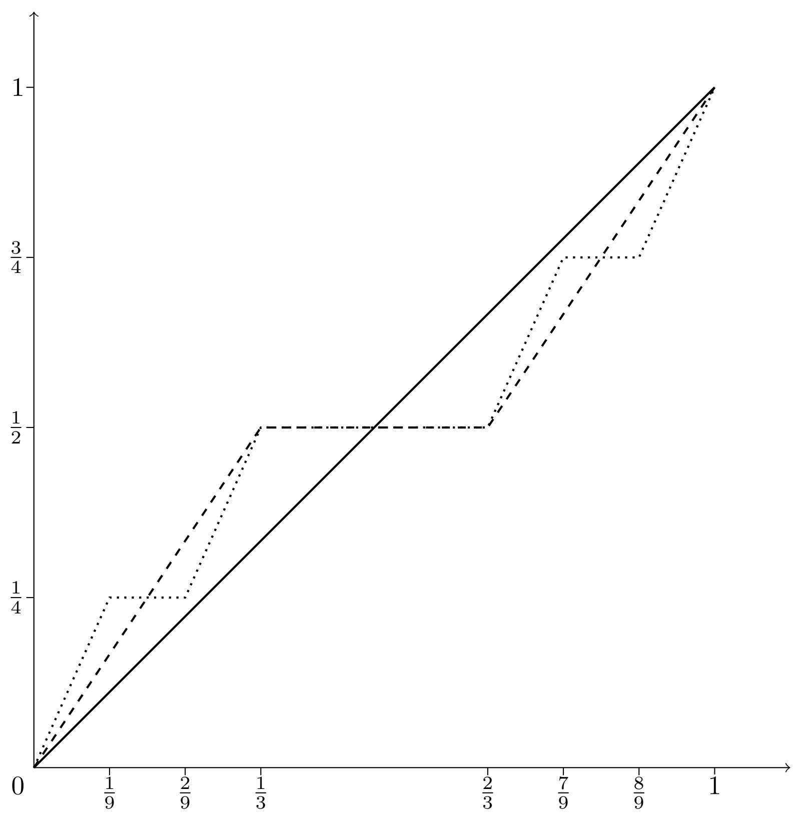
Theorem 25: Let $A \subset \mathbb{R}$ and let (f_n) be a sequence of *continuous* functions $f_n : A \to \mathbb{R}$ that converges *uniformly* to $f : A \to \mathbb{R}$. Then f is continuous.

Proof. ("3ε proof" or "$\frac{\varepsilon}{3}$ proof")
Fix $c \in A$. Fix any $\varepsilon > 0$ and use the uniform convergence to choose $N = N(\frac{\varepsilon}{3})$ s.t. $n \geq N(\frac{\varepsilon}{3})$, $x \in A \implies |f_n(x) - f(x)| \leq \frac{\varepsilon}{3}$. Use continuity of f_N at c to give $\delta > 0$ s.t. $|x - c| < \delta \implies |f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$. Then $|x - c| < \delta,\ x \in A \implies |f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.
Hence f is continuous at c. c is any point in A so f is continuous. □

7 Uniform Convergence: Construction of Exotic Examples

The Devil's Staircase/Cantor function: Let $f : [0,1] \to \mathbb{R}$, $n = 0,1,2,...$ be piecewise linear non-decreasing *continuous* functions defined recursively by:

$$f_0(x) = x,\ f_{n+1}(x) = \begin{cases} \frac{1}{2}f_n(3x), & 0 \le x \le \frac{1}{3} \\ \frac{1}{2}, & \frac{1}{3} \le x \le \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2}f_n(3x-2), & \frac{2}{3} \le x \le 1 \end{cases},\ n \ge 0$$



So $f_n(0) = 0,\ f_n(1) = 1\ \forall n$.
At each $n \ge 1$ add 2^{n-1} flat patches of length $(\frac{1}{3})^n$ slopes of linear bits: $(\frac{3}{2})^n$.

$$\sup_{x\in[0,1]} |f_{n+1}(x) - f_n(x)| \le \sup_{x\in[0,1]} |f_n(x) - f_{n-1}(x)|,\ n \ge 1$$

So $\|f_m - f_n\|_\infty \le (\frac{1}{2})^n \|f_1 - f_0\| \le \frac{1}{2}$

$$\begin{aligned} \text{If } n > m,\ \|f_m - f_n\|_\infty &\le \|f_m - f_{m+1}\|_\infty + \|f_{m+1} - f_{m+2}\|_\infty + ... + \|f_{n-1} - f_n\|_\infty \\ &\le \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + ... + \frac{1}{2^n} \\ &\le \frac{1}{2^{m+1}}\left(\frac{1}{1-\frac{1}{2}}\right) \\ &= \frac{1}{2^m} \end{aligned}$$

So (f_n) is uniformly Cauchy so converges uniformly to a continuous function f (the Devil's Staircase) by Theorem 25.

Remark: Let $E = (\frac{1}{3}, \frac{2}{3}) + (\frac{1}{9}, \frac{2}{9}) + (\frac{7}{9}, \frac{8}{9}) + (\frac{1}{27}, \frac{2}{27}) + ...$
f is locally constant on each subinterval of E so is differentiable $\forall x \in E$ with $f'(x) = 0$. Total length of $E : \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + ... = \frac{1}{3}\{1 + \frac{2}{3} + \frac{4}{9} + ... + (\frac{2}{3})^n + ...\} = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1!$
Thus f is continuous, has zero derivative on practically the whole of $[0, 1]$, yet manages to increase from 0 to 1. Clearly $f(1) - f(0) = 1 = \int_0^1 f'$ fails. f' at end points of flat patches?
e.g. at $x = \frac{2}{3}$:

$$\frac{f(\frac{2}{3}+h) - f(\frac{2}{3})}{h} = 0 \text{ if } -\frac{1}{3} < h < 0 \text{ but } \frac{f(\frac{2}{3}+h) - f(\frac{2}{3})}{h} = \frac{\frac{1}{4}}{\frac{1}{9}} = \frac{9}{4}$$

if $h = \frac{1}{9}$, $\frac{27}{8}$ if $h = \frac{1}{27}$, $(\frac{3}{2})^n$ if $h = \frac{1}{3^n}$. i.e.

$$\limsup_{h \searrow 0} \frac{f(\frac{2}{3}+h) - f(\frac{2}{3})}{h} = \infty$$

is *not* differentiable at end points.
Let $C = [0, 1] \setminus E$ (it is a Cantor set)
C is not empty (it contains the end points: $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, ...$) but *much more*: it is the set of point in $[0, 1]$ which can be expressed in base 3 without the digit 1.

8 Uniform Convergence and Integration

Some results for functions of two variables:
Let $D = [a,b] \times [c,d] \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be continuous. Recall continuity at $(x_0, t_0) \subset D$ means $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $(|x - x_0| < \delta,\ |t - t_0| < \delta,\ (x,t) \in D) \implies |f(x,t) - f(x_0,t_0)| < \varepsilon$ – continuous on D means continuous at all points of D.

Definition 10: $f : D \to \mathbb{R}$ is *uniformly continuous* if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $(|x-y| < \delta,\ |t-s| < \delta,\ x, y \in [a,b],$ $t, s \in [c,d]) \implies |f(x,t) - f(y,s)| < \varepsilon$. In particular $|f(x,t) - f(x,s)| < \varepsilon$ (put $y = x$) i.e. $f(\cdot,t) \to f(\cdot,s)$ uniformly as $t \to s$. Hence by Theorem 24, $\int_a^b f(x,t)\,dx \to \int_a^b f(x,s)\,dx$ as $t \to s$ (i.e $\lim \int = \int \lim$).

Recall: $f_n \to f$ uniformly, $(\int f_n) \to \int f$ as $n \to \infty$

Lemma 26: $f : D \to \mathbb{R}$ continuous implies f is uniformly continuous (see Theorem 13).

Proof. Repeat proof of Theorem 13 in 1 dimension (exercise). □

Note: Given $((x_n, t_n)) \in [a,b] \times [c,d]$:

1. $\exists$ subsequence $((x_{n_k}, t_{n_k}))$ s.t. $x_{n_k} \to x^*$ as $k \to \infty$
2. $\exists$ subsequence $((x_{n_{k_j}}, t_{n_{k_j}}))$ s.t. $t_{n_{k_j}} \to t^*$ as $j \to \infty$

Proposition 27 (differentiation under the integral): Let f, $\frac{\partial f}{\partial t}$ be continuous for $(x,t) \in [a,b] \times (c,d)$ where c, d may be finite or ∞. Then the functions F, G defined by $F(t) = \int_a^b f(x,t)\,dx$, $G(t) = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$ exist on (c,d) and F is differentiable with $F' = G$.

Proof. The integrals exist for $t \in (c,d)$ since f and $\frac{\partial f}{\partial t}$ be continuous w.r.t. x. Given $t \in (c,d)$ choose bounded and closed $[c_1, d_1] \subset (c,d)$ with $c_1 < t < d_1$. Now f and $\frac{\partial f}{\partial t}$ are continuous on $D_1 := [a,b] \times [c_1, d_1]$ hence uniformly continuous there. Hence given $\varepsilon > 0\ \exists \delta > 0$ s.t. $(|x - y| < \delta,\ |t - s| < \delta,\ x, y \in [a,b],$ $t, s \in [c_1, d_1]) \implies \left| \frac{\partial f}{dt}(x,t) - \frac{\partial f}{\partial t}(y,s) \right| < \varepsilon$. Then for $h \neq 0$ and $t + h \in [c_1, d_1]$ we have:

$$\left| \frac{F(t+h) - F(t)}{h} - G(t) \right| = \left| \int_a^b \left[\frac{f(x,t+h) - f(x,t)}{h} - \frac{\partial f}{\partial t}(x,t) \right] dx \right| = \left| \int_a^b \left[\frac{\partial f}{\partial t}(x,z) - \frac{\partial f}{\partial t}(x,t) \right] dx \right|$$

Using MVT: $f(x, t+h) - f(x,t) = h \frac{\partial f}{\partial t}(x,z)$ where z is between t and $t + h$.
Hence if $|h| < \delta$, $\left| \frac{F(t+h) - F(t)}{h} - G(t) \right| \le |b - a|\varepsilon$. $\therefore F$ is differentiable and:

$$\frac{d}{dt} \int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$$

□

Theorem 28 (Fubini/swap order of integration): Let h be *continuous* on $D = [a,b] \times [c,d]$. Then:

$$\int_a^b \left(\int_c^d h(x,y) dy \right) dx = \int_c^d \left(\int_a^b h(x,y)\,dx \right) dy$$

Proof. Set:

$$H(t) = \int_a^t \left(\int_c^d h(x,y) dy \right) dx - \int_c^d \left(\int_a^t h(x,y)\,dx \right) dy$$

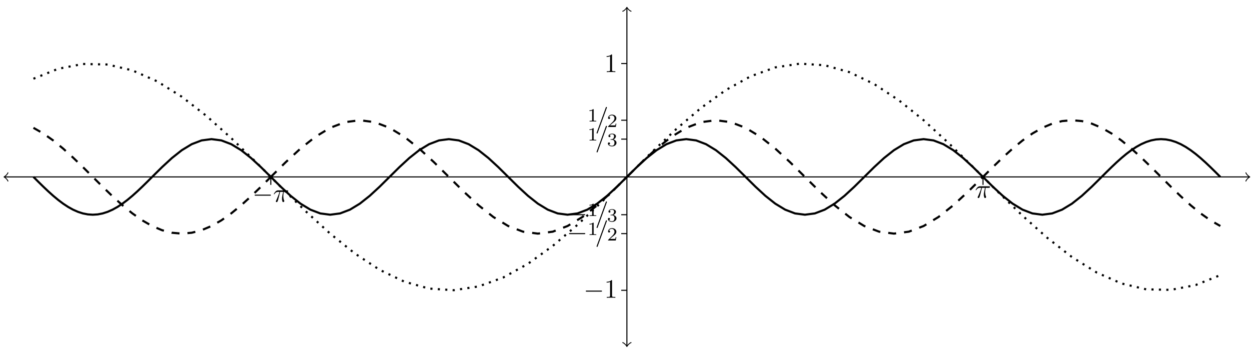
$H(a) = 0$ and:

$$\begin{aligned}
\frac{dH}{dt}(t) &= \int_c^d h(t,y)dy - \int_c^d \frac{\partial}{\partial t}\left(\int_a^t h(x,y)\,dx\right)dy && \text{(by the FTC and Proposition 27, respectively)} \\
&= \int_c^d h(t,y)dy - \int_c^d h(t,y)dy \\
&= 0 && (\forall t \in [a,b])
\end{aligned}$$

$\therefore H(t) = 0 \ \forall t \in [a,b]$, in particular for $t = b$ as required. □

9 Uniform Convergence: its Advantages for Differentiability

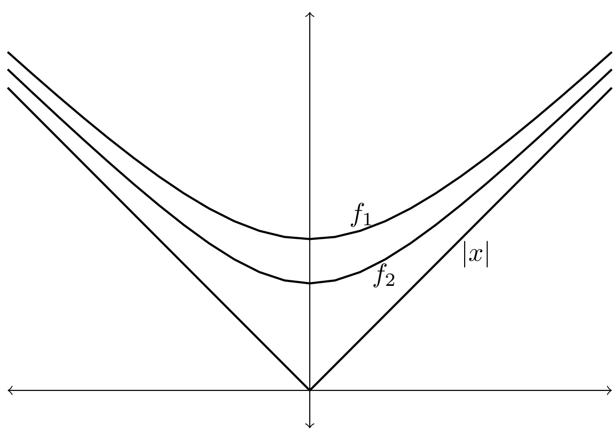
Example: $f_n, f : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n}\sin(nx)$, $f = 0$. Then $\|f_n - f\|_\infty = \frac{1}{n}$ (since $\|\sin\|_\infty = 1$) so $f_n \to f$ uniformly as $n \to \infty$. Each f_n is differentiable and $f_n'(x) = \cos nx$ so $f_n'(0) \to 1$ as $n \to \infty$ but $f_n'(x)$ does not converge if $x \neq 2\pi m$ e.g. $(f_n'(\pi)) = ((-1)^n)$. Although: f' is differentiable ($f' = 0$), we do *not* have $f_n' \to f'$:



Example: $f_n, f : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$. $f(x) = |x|$, f_n is differentiable with:

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}} \to \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}, \forall x \in \mathbb{R}$$

$|f_n(x) - f(x)| \leq |f_n(0) - f(0)| = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$ so f_n converges uniformly to f but is *not* differentiable.



Theorem 29: Let (f_n) be a sequence of $\mathcal{C}^1$ functions $[a,b] \to \mathbb{R}$ that converge *pointwise* to some function $f : [a,b] \to \mathbb{R}$. Suppose that (f_n') converges *uniformly* to a function g. Then f is $\mathcal{C}^1$ and $f' = g$.

Proof. Fix $x \in [a,b]$. Since (f_n') converges uniformly to g, g is continuous by Theorem 25 and Theorem 24 implies:

$$\left(\int_a^x f_n'\right) \to \int_a^x g \text{ as } n \to \infty$$

Now $\int_a^x f_n' = f_n(x) - f_n(a)$ by FTC 2 (Theorem 18) and $f_n(x) - f_n(a) \to f(x) - f(a)$ by pointwise convergence. Thus $\forall x \in [a,b]$, $f(x) - f(a) = \int_a^x g$. Then FTC 1 (Corollary 15, swap f and g) $\implies$ f is differentiable on $[a,b]$ with derivative g. Since g is continuous, f is $\mathcal{C}^1$. ☐

In the example above, Theorem 29 $\implies$ (f_n') cannot converge uniformly (check this directly!).

Theorem 30 (the Weierstrass M-test for uniform convergence): Let $f_k : A \to \mathbb{R}$ be functions. Suppose there are constants $M_k > 0$ such that $\forall x \in A$ $|f_k(x)| \leq M_k$ and $\sum_{k=1}^{\infty} M_k$ converges. Then $(\sum_{k=1}^{n} f_k)$ converges uniformly (to some function $f : A \to \mathbb{R}$).

Proof. $t_n := \sum_{k=1}^{n} M_k$ converges as $n \to \infty$ so (t_n) is a Cauchy sequence in $\mathbb{R}$ and $\forall \varepsilon > 0$ $\exists N(\varepsilon)$ such that $m \geq n \geq N(\varepsilon) \implies |t_m - t_n| < \varepsilon$. We show (s_n) is uniformly Cauchy, hence converges uniformly. For $m > n$ in $\mathbb{N}$, $|s_m(x) - s_n(x)| = |\sum_{k=n+1}^{m} f_k(x)| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} M_k = |t_m - t_n| < \varepsilon$ if $m \geq n \geq N(\varepsilon)$. □

Corollary 30: If the series $\sum_{k=0}^{\infty} \|f_k\|_\infty < \infty$ then $(\sum_{k=1} f_k)$ converges uniformly as $n \to \infty$.

Proof. Take $M_k = \|f_k\|_\infty$ for each k in Theorem 30. □

Theorem 31: If the series $\sum_{k=0}^{\infty} |a_k|$ and $\sum_{k=1}^{\infty} |b_k|$ converge then the Fourier series $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$ converges uniformly. The limit function, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\forall x \in \mathbb{R}$, $f(x + 2\pi) = f(x)$. Also,

$$\int_{-\pi}^{\pi} f(x) \cos kx \, dx = \pi a_k \ (\forall k \geq 0) \text{ and } \int_{-\pi}^{\pi} f(x) \sin kx \, dx = \pi b_k \ (\forall k \geq 1)$$

Proof. $\forall x \in \mathbb{R}$, $|a_k \cos kx + b_k \sin kx| \leq |a_k| + |b_k| =: M_k$
By Theorem 30, $s_n(x) := \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$ converges uniformly, to some function $f : \mathbb{R} \to \mathbb{R}$, which by Theorem 25 is continuous. Since $\forall x \in \mathbb{R}$, $s_n(x + 2\pi) = s_n(x)$ we have $f(x + 2\pi) = f(x)$

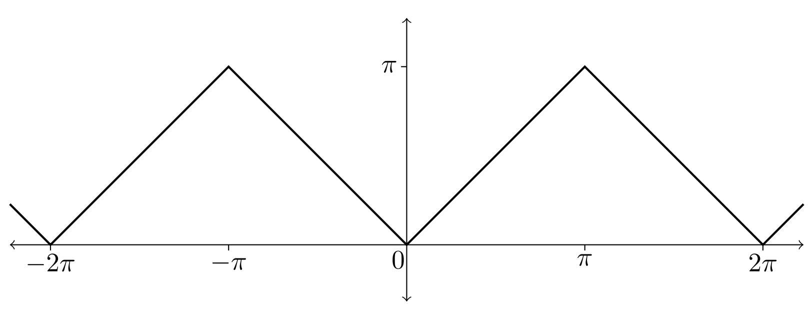
Recall:

$$\int_{-\pi}^{\pi} \cos^2 kx \, dx = \pi = \int_{-\pi}^{\pi} \sin^2 kx \, dx \ \forall k \geq 1$$
$$\int_{-\pi}^{\pi} \cos kx \cos lx \, dx = 0 = \int_{-\pi}^{\pi} \sin kx \sin lx \, dx \text{ if } k \neq l$$
$$\int_{-\pi}^{\pi} \cos kx \sin lx \, dx = 0 \ \forall k, l$$

Hence if $n \geq k$ then $\int_{-\pi}^{\pi} s_n(x) \cos kx \, dx = a_k \pi$ $(\forall k \geq 0)$ $\left(\int_{-\pi}^{\pi} \frac{a_0}{2} \, dx = a_0 \pi\right)$ and $\int_{-\pi}^{\pi} s_n(x) \sin kx = b_k \pi$ $(\forall k \geq 1)$.
So Theorem 24', $\int_{-\pi}^{\pi} f(x) \cos kx \, dx = a_k \pi$ $(\forall k \geq 0)$, $\int_{-\pi}^{\pi} f(x) \sin kx \, dx = b_k \pi$ $(\forall k \geq 1)$ □

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ for $|x| \leq \pi$ and $\forall x \in \mathbb{R}$, $f(x) = f(x + 2\pi)$.



$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx \\ &= \pi \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos kx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos kx\,dx = \frac{2}{\pi}\left[\frac{x\sin kx}{k}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin kx}{k}\,dx = \frac{2}{\pi k^2}\left[\cos kx\right]_0^{\pi} = \frac{-2}{\pi k^2}(1-(-1)) \\ &= -\frac{4}{\pi k^2} \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin kx\,dx \\ &= 0 \end{aligned}$$

$$\sum_{k=1}^{\infty} |a_k| = \left|-\frac{4}{\pi}\right|\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2} < \infty$$

so:

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2j+1)x}{(2j+1)^2}$$

converges uniformly to $|x|$ on $[-\pi, \pi]$.

2. $f(x) = x$, $-\pi \le x < \pi$ and $f(x) = f(x + 2\pi)$.

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx\,dx \\ &= 0 \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx\,dx \\ &= -\frac{2}{k}(-1)^k \end{aligned}$$

Now:

$$\sum_{k=1}^{\infty} |b_k| = 2\sum_{k=1}^{\infty}\frac{1}{k}$$

diverges so the Fourier series does not converge uniformly but converges pointwise.

Theorem 32 (the Riemann zeta function): the series $\sum_{k=1}^{\infty}\frac{1}{k^x}$ converges pointwise for $x > 1$ and $\zeta : (1, \infty) \to \mathbb{R}$, $\zeta(x) := \sum_{k=1}^{\infty}\frac{1}{k^x}$ is continuous.

Proof. Fix $a > 1$. $\forall n \in \mathbb{N}$,

$$\sum_{k=2}^{n}\frac{1}{k^a} < \int_1^n \frac{1}{t^a}\,dt = \left[\frac{t^{1-a}}{1-a}\right]_1^n \to \frac{1}{a-1} \text{ as } n \to \infty$$

If $M_k = \frac{1}{k^a}$ then $\sum_{k=1}^{\infty} M_k$ converges and $\forall x \in [a, \infty)$, $\frac{1}{k^x} \le M_k$ so by Weierstrass M-test (Theorem 30), $\sum_{k=1}^{\infty}\frac{1}{k^x}$ converges *uniformly* on $[a, \infty)$. By Theorem 25' its limit ζ is continuous on $[a, \infty)$. This holds $\forall a > 1$. So $\zeta : (1, \infty)$ is continuous. □

Note:

1. On $(1, \infty)$ it is *not* true that $\sum_{k=1}^{\infty}\frac{1}{k^x}$ converges uniformly because as $x \to 1$

$$|s_{2n}(x) - s_n(x)| = \sum_{k=n+1}^{2n}\frac{1}{k^x} \to \sum_{k=n+1}^{2n}\frac{1}{k} > \frac{n}{2n} = \frac{1}{2}$$

so it is false that $\forall x \in (1, \infty)$. $\sum_{k=n+1}^{2n}\frac{1}{k^x} < \frac{1}{4}$, say. So *not* uniformly Cauchy, therefore not uniformly convergent.

2.
$$\frac{d}{dx}\frac{1}{k^x} = -\log k \frac{1}{k^x} \text{ and } \sum_{k=2}^{\infty} \log k \frac{1}{k^a} < \infty \text{ for } a > 1.$$

So Theorem 24' gives ζ is $\mathcal{C}^1$ on $[a, \infty)$ hence on $(0, 1)$. In fact ζ is $\mathcal{C}^\infty$.

Theorem 33 (a nowhere differentiable curve): There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ with the property that $\forall x \in \mathbb{R}$, f is *not* differentiable at x.

Proof. Define $g : \mathbb{R} \to \mathbb{R}$ by $|x|$ for $x \in [-1, 1]$ and $\forall x \in \mathbb{R}$, $g(x + 2) = g(x)$
Note that

$$\begin{aligned} |g(x)| &\leq 1 && \forall x \in \mathbb{R} \\ |g(x) - g(y)| &\leq |x - y| && \forall x, y \end{aligned}$$

So g is (uniformly) continuous (put $\delta = \varepsilon$)
$\forall k \in \mathbb{Z}_+$ let $f_k : \mathbb{R} \to \mathbb{R}$, $f_k(x) = (\frac{3}{4})^k g(4^k x)$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{k=0}^{\infty} f_k(x)$
$\forall x \in \mathbb{R}$, $|f_k(x)| \leq (\frac{3}{4})^k =: M_k$ and M_k converges to $\frac{1}{1-\frac{3}{4}} = 4$.
Thus by Weierstrass M-test (Theorem 30) the series converges uniformly so f is continuous.
Fix $x \in \mathbb{R}$ and $m \in \mathbb{N}$. Put $h_m = \pm\frac{1}{2}4^{-m}$ where the sign is chosen so that there are no integers in the open interval of length $\frac{1}{2}$ from $4^m x$ to $4^m(x + h_m)$. Let:

$$\gamma_k := \frac{f_k(x + h_m) - f_k(x)}{h_m} = \left(\frac{3}{4}\right)^k \left(\frac{g(4^k(x \pm \frac{1}{2}4^{-m})) - g(4^k x)}{\pm\frac{1}{2}4^{-m}}\right)$$

If $k > m$ then $\frac{4}{2}^{k-m}$ is an integer so $\gamma_k = 0$ by periodicity. For $0 \leq k \leq m$ gives $|\gamma_k| \leq 3^k$. Now $|\gamma_m| = 3^m$ so we have:

$$\left|\frac{f(x + h_m) - f(x)}{h_m}\right| = \left|\sum_{k=0}^{\infty}\gamma_k\right| = \left|\sum_{k=0}^{m}\gamma_k\right| = \left|3^m + \sum_{k=0}^{m-1}\gamma_k\right| \geq 3^m - \sum_{k=0}^{m-1} 3^k = \frac{1}{2}(3^m + 1) \text{ as } m \to \infty,\ h_m \to 0$$

□

10 Series of Functions

Many useful functions can only be defined by approximations by elementary functions. A power series is a limit of polynomials (needed for exp, sin, cos,...). A Fourier series is a limit of differentiable functions of period 2π:

$$\frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

Let $f_k : A \to \mathbb{R}$ $(k = 1, 2, ...)$ be functions. Form the partial sums $s_n : A \to \mathbb{R}$, $s_n(x) = \sum_{k=1}^{n} f_k(x)$. Say the series $\sum_k f_k$ *converges uniformly* (or pointwise) to a function $f : A \to \mathbb{R}$ if (s_n) converges uniformly (or pointwise) to f. We get immediately:

Theorem 24': If $A = [a, b]$, $\forall k \in \mathbb{N}$ f_k is regulated and $(s_n) \to f$ uniformly then f is regulated and:

$$\left(\sum_{k=1}^{n} \int_a^b f_k\right) \to \int_a^b f \text{ as } n \to \infty$$

Proof. Use Proposition 11 (linearity of I) and Theorem 24. □

Theorem 25': If $\forall k \in \mathbb{N}$ $f_k : A \to \mathbb{R}$ is continuous and $(s_n) \to f$ uniformly then $f : A \to \mathbb{R}$ is continuous.

Proof. Use $\sum_{k=1}^{n}$ continuous functions and Theorem 25 □

Theorem 29': If $A = [a, b]$ $\forall k \in \mathbb{N}$, f_k is $\mathcal{C}^1$, $(s_n) \to f$ pointwise and (s_n') converges uniformly to some $g : [a, b] \to \mathbb{R}$ then f is $\mathcal{C}^1$ and $f' = g$.

Proof. Use $\left(\sum^n f_k\right)' = \sum_1^n f_k'$ and Theorem 29. □

11 Normed Vector Spaces

Definition 11: Let V be a real vector space. A *norm* on V is a function $V \to \mathbb{R}$, written $v \mapsto \|v\|$, satisfying:

(i) $\forall v \in V$, $\|v\| \geq 0$; $\|v\| = 0 \Leftrightarrow v = 0_V$

(ii) $\forall \lambda \in \mathbb{R}, \forall v \in V$, $\|\lambda v\| = |\lambda| \|v\|$

(iii) $\forall v, v' \in V$, $\|v + v'\| \leq \|v\| + \|v'\|$ (triangle inequality)

A *normed vector space* is a pair $(V, \|\cdot\|)$, where $\|\cdot\|$ is a norm on the vector space V.

Examples:

1. $|\cdot|$ is a norm on $\mathbb{R}$.

2. The sup norm on $\mathcal{B}[a,b]$, $\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|$ is a norm.

Proof. $\|f\|_\infty = 0 \implies \forall x \in [a,b], |f(x)| = 0 \implies f = 0$
$\|f+g\|_\infty = \sup_x(|f(x)+g(x)|) \leq \sup_x(|f(x)|+|g(x)|) \leq \sup_x |f(x)| + \sup_x |g(x)| = \|f\|_\infty + \|g\|_\infty$ □

Proposition 34 (norms on $\mathbb{R}^n$): On $\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_j \in \mathbb{R}, 1 \leq k \leq n\}$:

$$\|\mathbf{x}\|_1 := \sum_{j=1}^n |x_j|,\ \|\mathbf{x}\|_2 := \sqrt{\sum_{j=1}^n |x_j|^2} \text{ and } \|\mathbf{x}\|_\infty := \max_{1\leq j\leq n} |x_j| \text{ are norms.}$$

Remarks:

1. These are cases $p = 1, 2, \lim_{p\to\infty}$ of $\|\mathbf{x}\|_p := \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}}$

   (The proof of (iii) needs Minkowski inequality: $\left(\sum_{j=1}^n |x_j + y_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p\right)^{\frac{1}{p}}$)

2. If you think of $(x_1, x_2, ..., x_n)$ as the image of $\mathbf{x} : (1, 2, ..., n) \to \mathbb{R}$ then $\|\mathbf{x}\|_\infty$ corresponds to $\|f\|_\infty$ on $\mathcal{B}[a,b]$.

Proof of Proposition 34. (i), (ii) are easily checked for each norm (exercise). Taking $\sum_{j=1}^n$ or $\max_{i\leq j\leq n}$ of $|x_j + y_j| \leq |x_j| + |y_j|$ gives (iii) for $\|\cdot\|$, and $\|\cdot\|_\infty$. To show (ii) for $\|\cdot\|_2$, use Cauchy-Schwartz inequality in $\mathbb{R}^n$: $\sum_{j=1}^n a_j b_j \leq \left(\sum_{j=1}^n a_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n b_j^2\right)^{\frac{1}{2}}$

Proof of Cauchy-Schwartz inequality. $(a_1 + \lambda b_1)^2 + (a_2 + \lambda b_2)^2 + ... + (a_n + \lambda b_n)^2 \geq 0$. So at most one real root in λ. Thus discriminant ≤ 0, i.e.

$$\left(\sum_{j=1}^n a_j b_j\right)^2 \leq \left(\sum_{j=1}^n a_j^2\right)\left(\sum_{j=1}^n b_j^2\right)$$

□

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|_2^2 &= \sum_{j=1}^{n}(x_j+y_j)^2 \\
&= \sum_{j=1}^{n} x_j(x_j+y_j) + \sum_{j=1}^{n} y_j(x_j+y_j) \\
&\leq \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n}(x_j+y_j)^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} y_i^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n}(x_j+y_j)^2\right)^{\frac{1}{2}} \\
&\leq \|\mathbf{x}\|_2\|\mathbf{x}+\mathbf{y}\|_2 + \|\mathbf{y}\|_2\|\mathbf{x}+\mathbf{y}\|_2, \text{ hence } \|\mathbf{x}+\mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2
\end{aligned}$$

□

Definition 12: The norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a vector space V are called *equivalent* if $\exists K_1 > 0$, $K_2 > 0$ s.t. $\forall v \in V$:

$$K_1\|v\|_a \leq \|v\|_b \leq K_2\|v\|_a$$

This is clearly an equivalence relation on the set of norms on V'.

Lemma 35: $\forall n \in \mathbb{N}$ the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$ are equivlent.

Proof. Exercise. □

Definition 13: In a normed vector space $(V, \|\cdot\|)$ we say that a sequence $(y_n)_{n=1}^{\infty}$ converges to $y \in V$ if $\|y_n - y\| \to 0$ as $n \to \infty$, that is if $\forall \varepsilon > 0\ \exists N = N(\varepsilon) \in \mathbb{N}$ s.t. $n \geq N \implies \|y_n - y\| < \varepsilon$. In $(V, \|\cdot\|)$ the sequence $(y_n)_{n=1}^{\infty}$ is said to be Cauchy if $\forall \varepsilon > 0\ \exists M = M(\varepsilon)$ s.t. $m \geq n \geq M \implies \|y_m - y_n\| < \varepsilon$. If every Cauchy sequence in $(V, \|\cdot\|)$ converges (to some point in V) then say $(V, \|\cdot\|)$ is a *Banach space*. If $A \subset V$ and every Cauchy sequence of elements of A converges in $(V, \|\cdot\|)$ to an element of A, say that A is *complete*. So a *Banach space* is a *complete normed vector space*.

Note:

1. If $(y_n) \to y$ in $(V, \|\cdot\|)$ as $n \to \infty$ then (y_n) is a Cauchy sequence (use $N(\frac{\varepsilon}{2})$ for $M(\varepsilon)$).
2. If $(y_n) \to y$ and $(y_n) \to z$ then $\|y - z\| \leq \|y - y_n\| + \|y_n - z\| \to 0$. So $\|y - z\| = 0$ and so $y - z = 0_V$. Thus $y = z$ (uniqueness of limit).
3. $\|\cdot\|_\infty$ is a norm on vector space $\mathcal{S}[a,b]$ of step functions but this is *not* a Banach space because a regulated function $f \in \mathcal{R}[a,b] \setminus \mathcal{S}[a,b]$ has a sequence $(\varphi_n) \to f$, $\|\varphi_n - f\|_\infty \to 0$ with $f \notin \mathcal{S}[a,b]$.
4. $(\mathcal{R}[a,b], \|\cdot\|_\infty)$ *is* a Banach space by Theorem 24.
5. $(\mathbb{R}, |\cdot|)$ is a 1-dimensional Banach space. In this $[0,1]$ is complete but $(0,1)$ is *not* complete ($\frac{1}{n} \to 0 \notin (0,1)$).
6. If $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms of V then $(y_n) \to y$ in $(V, \|\cdot\|_a) \Leftrightarrow y_n \to y$ in $(V, \|\cdot\|_b)$. Thus $(V, \|\cdot\|_a)$ is Banach $\Leftrightarrow (V, \|\cdot\|_b)$ is Banach.

Theorem 36: Any norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to $\|\cdot\|_\infty$.

Proof. Let $\mathbf{x} = (x_1, x_2, ..., x_n) = \sum_{j=1}^{n} x_j\mathbf{e}_j$ where $\{\mathbf{e}_j : 1 \leq j \leq n\}$ is a basis for $\mathbb{R}^n$

$$\|\mathbf{x}\| = \left\|\sum_{j=1}^{n} x_j\mathbf{e}_j\right\| \leq \sum_{j=1}^{n} |x_j|\|\mathbf{e}_j\| \leq \sum_{j=1}^{n} \|\mathbf{e}_j\|\|\mathbf{x}\|_\infty \qquad (*)$$

Let $J := \inf\left\{\frac{\|\mathbf{x}\|}{\|\mathbf{x}\|_\infty} : \mathbf{x}\in\mathbb{R}^n\setminus\{0\}\right\} = \inf\{\|\mathbf{x}\| : \|\mathbf{x}\|_\infty = 1\}$
(since (ii): $\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ $\forall\lambda\in\mathbb{R}$ so $\forall\mathbf{x}\neq\mathbf{0}$, $\frac{\|\mathbf{x}\|}{\|\mathbf{x}\|_\infty} = \frac{\|\lambda\mathbf{x}\|}{\|\lambda\mathbf{x}\|_\infty}$) It remains to show that $J > 0$ so that $J\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|$ $\forall\mathbf{x}\neq\mathbf{0}$. Suppose not, i.e. suppose $J = 0$. Take $(\mathbf{x}^k)_{k=1}^\infty \leq \mathbb{R}^n$ with $\|\mathbf{x}^k\|_\infty = 1$ and $\|\mathbf{x}^n\| < \frac{1}{k}$. The cube in $\mathbb{R}^n$ has $2n$ faces and there exists a subsequence of $(\mathbf{x}^k)$ s.t. all elements have $x_j^k = 1$ (or -1) for some $j\in\{1,...,n\}$. Take a subsequence of this subsequence with $x_1^k \to y_1$ (by Bolzano-Weierstrass Theorem) then a subsequence of this with $x_2^k \to y_2, ...,$ then a subsequence of this with $x_n^k \to y_n$. So the final subsequence tends to $\mathbf{y} = (y_1, y_2, ..., y_n)$. Now $|y_j| = 1$ or -1 and $\|\mathbf{y}\| > 0$. Then $0 < \|\mathbf{y}\| = \|\mathbf{y}-\mathbf{x}^k+\mathbf{x}^k\| \leq \|\mathbf{y}-\mathbf{x}^k\| + \|\mathbf{x}^k\| \leq \left(\sum_{j=1}^n \|\mathbf{e}_j\|\right)\|\mathbf{x}^k-\mathbf{y}\|_\infty + \frac{1}{k}$ (by $(*)$)$= \frac{1}{k} + \left(\sum_{j=1}^n \|\mathbf{e}_j\|\right)\left(\max_{1<j\leq n}|x_j^k - y_j|\right) \to 0$ along the final subsequence. □

Proposition 37: $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a Banach space (and hence so is any $(\mathbb{R}^n, \|\cdot\|)$).

Proof. If $(x_j^k)_{k=1}^\infty$ in $\mathbb{R}^n$ is Cauchy then for each $1\leq j\leq n$, $|x_j^k - x_j^l| \leq \max_{1\leq j\leq n}|x_j^k - x_j^l| =: \|x^k - x^l\|_\infty$ so $(x_j^k)_{k=1}^\infty$ is Cauchy in $(\mathbb{R}, |\cdot|)$ and $x_j^k \to a_j$. Then $x^k \to (a_1, a_2, ..., a_n)$ in $(\mathbb{R}^n, \|\cdot\|_\infty)$. □

Proposition 38: Let $-\infty < a < b < \infty$.

1. The following are norms on $\mathcal{C}[a,b] := \{f : [a,b]\to\mathbb{R} : f \text{ is continuous}\}$:
   (a) $\|f\|_\infty := \max_{a\leq x\leq b}|f(x)|$
   (b) $\|f\|_1 := \int_a^b |f(x)|\,dx$
   (c) $\|f\|_2 := \sqrt{\int_a^b |f(x)|^2\,dx}$
2. $(\mathcal{C}[a,b], \|\cdot\|_\infty)$ is a Banach space. $(\mathcal{C}[a,b], \|\cdot\|_1)$, $(\mathcal{C}[a,b], \|\cdot\|_2)$ are *not* Banach spaces.
3. On $\mathcal{C}[0,1]$, $\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty$ (†)
4. $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are *not* equivalent norms on $\mathcal{C}[a,b]$.

Remark: Compare (†) with $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$ in $\mathbb{R}^n$.

Proof.

1. (a) Done before.
   (b) (i) $\|f\|_1 \geq 0$ easy; $f = 0 \implies \|f\|_1 = 0$ easy. To show $\|f\|_1 = 0 \implies f = 0$: suppose $\|f\|_1 = 0$ and $f\neq 0$. Then $\exists t_0\in[a,b]$ where $f(t_0)\neq 0$. Now f is continuous at t_0 so $\exists$ a neighbourhood $[t_1,t_2]\subset[a,b]$ of t_0 s.t. $|f(t)| \geq \frac{1}{2}|f(t_0)|$ $\forall t\in[t_1,t_2]$. Then $\|f\|_1 = \int_a^b |f(t)|\,dt \geq \int_{t_1}^{t_2}|f(t)|\,dt \geq (t_2-t_1)\frac{1}{2}|f(t_0)| > 0$.
   (ii) $\|\lambda f\|_1 = |\lambda|\|f\|_1$ easy to prove.
   (iii) $\|f+g\|_1 = \int_a^b |f(x)+g(x)|\,dx \leq \int_a^b(|f(x)|+|g(x)|)\,dx = \|f\|_1 + \|g\|_1$
   (c) (i) The same as the one for (b).
   (ii) $\|\lambda f\|_2 = |\lambda|\|f\|_2$ easy to prove.
   (iii) $\|f+g\|_2^2 = \int_a^b(f+g)^2 = \int_a^b f(f+g) + \int_a^b g(f+g) \leq \left(\int_a^b f^2\right)^{\frac{1}{2}}\left(\int_a^b(f+g)^2\right)^{\frac{1}{2}} + \left(\int_a^b g^2\right)^{\frac{1}{2}}\left(\int_a^b(f+g)^2\right)^{\frac{1}{2}}$ by the Cauchy-Schwarz inequality. Using exercise 3B: $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$
2. Every Cauchy sequence converges in $(\mathcal{C}[a,b], \|\cdot\|_\infty)$ because the uniform limit of continuous functions is continuous (by Theorem 25) and in normed vector spaces a convergent sequence is Cauchy (see the first note on the previous page). $\therefore$ $(\mathcal{C}[a,b], \|\cdot\|_\infty)$ complete hence Banach.

   $(\mathcal{C}[a,b], \|\cdot\|_1)$ is *not* complete, e.g. WLOG $[a,b] = [0,1]$. (f_k) *is* Cauchy in $(\mathcal{C}[0,1], \|\cdot\|_1)$ because $\forall l > k$, $\int_0^1 |f_k - f_l| = \frac{1}{2}(\frac{1}{k} - \frac{1}{l}) < \varepsilon$ $\forall l, k \geq \frac{1}{2\varepsilon}$ but (f_k) does not converge to a function *in* $\mathcal{C}[0,1]$ because if f s.t. $f_k \to f$ is continuous then $\forall t < \frac{1}{2}$, $f(t) = 0$ (else if $f(t_0)\neq 0$ for some $t_0\in[0,\frac{1}{2})$ then there exists a neighbourhood $[t_1, t_2] \subset [0, \frac{1}{2})$ of t_0 where $|f(t)| \geq \frac{1}{2}|f(t_0)|$ so $\int_0^1 |f_k - f|dt \geq (t_2 - t_1)\frac{1}{2}|f(t_0)| > 0$ $\forall k > \frac{1}{\frac{1}{2} - t_0}$, so does not tend to zero as $k \to \infty$). Same argument shows that $f(t) = 1$, $\forall t > \frac{1}{2}$. Therefore f is discontinuous at $\frac{1}{2}$. Therefore $(\mathcal{C}[0,1], \|\cdot\|_1)$ is *not* complete hence not Banach. Exercise: $(\mathcal{C}[a,b], \|\cdot\|_2)$ is *not* complete hence not Banach.

Note: the sequence (f_k) above is *not* Cauchy in $(\mathcal{C}[0,1], \|\cdot\|_\infty)$. $\|f_k - f_l\| = \max_{x\in[0,1]} |f_k(x) - f_l(x)| = 1 - \frac{k}{l} \not\to 0$

3. $\|f\|_1 = \int_a^b |f| = \int_a^b |f| \cdot 1 \leq$ (by C-S) $\left(\int_a^b f^2\right)^{\frac{1}{2}} \left(\int_a^b 1\right)^{\frac{1}{2}} \leq \|f\|_2\sqrt{b-a}$ so $\|f\|_1 \leq \sqrt{b-a}\|f_2\|$. Now $\int_a^b f^2 \leq \|f^2\|_\infty (b-a)$ so $\|f\|_2 \leq \|f\|_\infty \sqrt{b-a}$

4. Consider $f_n : [0,1] \to \mathbb{R} : f_n(x) = \begin{cases} 1 - nx, & x \in \left[0, \frac{1}{n}\right] \\ 0, & x \in \left(\frac{1}{n}, 1\right] \end{cases}$
   Take $b = 1, a = 0$. Then $\|f_n\|_\infty = 1$ $\forall n$ and $\|f_n\|_1 = \frac{1}{2n}$. So $\nexists k$ s.t. $k\|f_n\|_\infty \leq \|f_n\|_1$ $\forall n \in \mathbb{N}$. $\therefore \|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent in $\mathcal{C}[a,b]$.

□

Definition 14: If $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ are normed vector spaces then a map $f : V \to W$ is said to be $(\|\cdot\|_V, \|\cdot\|_W)$*-continuous* or just *continuous* at $x \in V$ if $\forall \varepsilon > 0$ $\exists \delta = \delta_x(\varepsilon) > 0$ s.t. $\|x - y\|_V < \delta \implies \|f(x) - f(y)\|_W < \varepsilon$. If f is continuous at each $x \in V$ we say f is continuous.

Theorem 39: Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces and $T : V \to W$ be linear. Then the following are equivalent:

1. T is continuous at 0_V.
2. T is continuous.
3. $\{\|T(v)\|_W : \|v\|_V \leq 1\}$ is bounded on $\mathbb{R}$ (in which case we say T is *bounded*).

Proof.

$1 \implies 2$: Assume T is continuous at 0_V and fix any $v \in V$. $\forall \varepsilon > 0$ $\exists \delta = \delta_{0_V}(\varepsilon) > 0$ s.t. $\|v - 0_V\|_V = \|v\|_V < \delta \implies \|T(v) - T(0_V)\|_W = \|T(v) - 0_W\|_W = \|T(v)\|_W < \varepsilon$. If $\|y - v\|_V < \delta$ then $\|T(y) - T(v)\|_W = \|T(y - v)\|_W < \varepsilon$. Thus T is continuous.

$2 \implies 1$: Obvious.

$1 \implies 3$: T is continuous at 0_V so $\exists \delta(1)$ s.t. $\|v\|_V < \delta \implies \|T(v)\|_W < 1$. Then $\|u\|_V \leq 1 \implies \|\frac{1}{2}\delta u\|_V \leq \frac{1}{2}\delta < \delta \implies \|T(\frac{1}{2}\delta u)\|_W < 1 \implies \|T(u)\|_W \leq \frac{2}{\delta}$. Then $\sup\{\|T(u)\|_W : \|u\|_V \leq 1\} \leq \frac{2}{\delta}$ so T is bounded.

$3 \implies 1$: Put $K := \sup\{\|T(u)\|_W : \|u\|_V \leq 1\}$. Then $\forall \varepsilon > 0$, $\|v\|_V \leq \frac{\varepsilon}{2K} \implies \|\frac{2K}{\varepsilon}v\|_V \leq 1 \implies \frac{2K}{\varepsilon}\|T(v)\|_W = \|T(\frac{2K}{\varepsilon}v)\|_W \leq K \implies \|T(v)\|_W \leq \frac{\varepsilon}{2} < \varepsilon$. So putting $\delta_{0_V}(\varepsilon) = \frac{\varepsilon}{2K}$ we have T continuous at 0_V.

□

Proposition 40:

1. If $\|\cdot\|_V, \|\cdot\|'_V$ are equivalent norms on V and $\|\cdot\|_W, \|\cdot\|'_W$ are equivalent norms on W then a *linear* map $T : V \to W$ is $(\|\cdot\|_V, \|\cdot\|_W)$-continuous $\implies$ T is $(\|\cdot\|'_V, \|\cdot\|'_W)$-continuous.
2. Any linear map $T : \mathbb{R}^n \to W$ is $(\|\cdot\|_1, \|\cdot\|_W)$-continuous.
3. For any norm $\|\cdot\|$ on $\mathbb{R}^n$ any linear map $T : \mathbb{R}^n \to W$ is $(\|\cdot\|, \|\cdot\|_W)$-continuous.

4. $T : \mathcal{C}[0,1] \to \mathbb{R}$, $T(f) = f(0)$ is linear and $(\|\cdot\|_\infty, |\cdot|)$-continuous but *not* $(\|\cdot\|_1, |\cdot|)$-continuous.

Proof.

1. If $\forall v \in V$, $\forall w \in W$, $\|v\|_V \leq K\|v\|'_V$ and $\|w\|'_W \leq L\|w\|_W$ then $\|v\|'_V < \frac{1}{K}\delta(\frac{\varepsilon}{L}) \implies \|v\|_V \leq K\|v\|'_V < \delta(\frac{\varepsilon}{L}) \implies \|T(v)\|'_W \leq L\|T(v)\|_W < L \cdot \frac{\varepsilon}{L} = \varepsilon$

2. $\sum_{j=1}^n |x_j| = \|x\|_1 \leq 1 \implies \|T(x)\|_W = \|T(x_1\mathbf{e}_1 + ... + x_n\mathbf{e}_n)\|_W \leq \sum_{j=1}^n |x_j|\|T(\mathbf{e}_j)\|_W \leq \max_j \|T(\mathbf{e}_j)\|_W$ so T is bounded hence continuous.

3. Follows from 1. and 2. by equivalence of norms on $\mathbb{R}^n$ (Theorem 36 + Lemma 35).

4.
   - $T(\lambda f + \mu g) = \lambda f(0) + \mu g(0) = \lambda T(f) + \mu T(g)$
   - $\sup\{|f(x)|\} = \|f\|_\infty \leq 1 \implies |f(0)| \leq 1$ so $\sup\{|f(0)| : \|f\|_\infty \leq 1\} \leq 1$ and T is bounded hence continuous.
   - However consider $f_n(x) := \begin{cases} n - n^2x, & 0 \leq x \leq \frac{1}{n} \\ 0 & x > \frac{1}{n} \end{cases}$ then $\|f_n\|_1 = \int_0^1 |f_n| = \frac{1}{2} \leq 1$ but $\mathbb{N} = \{|f_n(0)| : n \in \mathbb{N}\} \subset \{f(0) : f \in \mathcal{C}[0,1], \|f\|_1 \leq 1\}$ is not bounded, hence T is *not* continuous.

□

Theorem 41: $L(V,W) := \{T : V \to W : T \text{ linear and } (\|\cdot\|_V, \|\cdot\|_W)\text{-continuous}\}$ is a vector space and $\|T\| := \sup\{\|T(v)\|_W : \|v\|_V \leq 1\}$ is a norm on it. It is called the *operator norm*.

Proof. $\lambda, \mu \in \mathbb{R}$, $S, T \in L(V,W) \implies \lambda T + \mu S$ is linear (Linear Algebra).

$$\begin{aligned} \|\lambda S + \mu T\| :&= \sup\{\|(\lambda S + \mu T)(v)\|_W : \|v\|_V \leq 1\} \\ &= \sup\{\|(\lambda S(v) + \mu T(v))\|_W : \|v\|_V \leq 1\} \\ &\leq \sup\{|\lambda|\|S(v)\|_W + |\mu|\|T(v)\|_W : \|v\|_V \leq 1\} \\ &\leq |\lambda| \sup_{\|v\|_V \leq 1} \|S(v)\|_W + |\mu| \sup_{\|v\|_V \leq 1} \|T(v)\|_W \\ &= |\lambda|\|S\| + |\mu|\|T\| < \infty \qquad (*) \end{aligned}$$

(iii): So $\lambda S + \mu T$ is bounded and so continuous (by Theorem 39). Hence $\lambda S + \mu T \in L(V,W)$. Putting $\lambda = \mu = 1$ in $(*)$ gives $\|S + T\| \leq \|S\| + \|T\|$

(ii): Also $\|\lambda S\| = \sup_{\|v\|_V \leq 1} \|\lambda S(v)\|_W = |\lambda| \sup_{\|v\|_V \leq 1} \|S(v)\|_W = |\lambda|\|S\|$

(i): Always $\|T\| \geq 0$ and $\|T\| = 0 \Leftrightarrow \sup\{\|T(v)\|_W : \|v\| \leq 1\} = 0 \Leftrightarrow (\|v\|_V \leq 1 \implies T(v) = 0_W) \Leftrightarrow \forall v \in V \setminus \{0_V\}$, $T(v) = \|v\|_V \cdot T(\frac{v}{\|v\|_V}) = \|v\|_V \cdot 0_W$ since $\|\frac{v}{\|v\|_V}\|_V = 1 = 0_W$. $v = 0_V \implies T(v) = 0_W \Leftrightarrow T = 0_{L(V,W)}$

□

Remark: The set of $m \times n$ matrices over $\mathbb{K}$, $L(\mathbb{R}^n, \mathbb{R}^m) \cong$ vector space of all these are bounded by Proposition 40 part 3.

Definition 15:

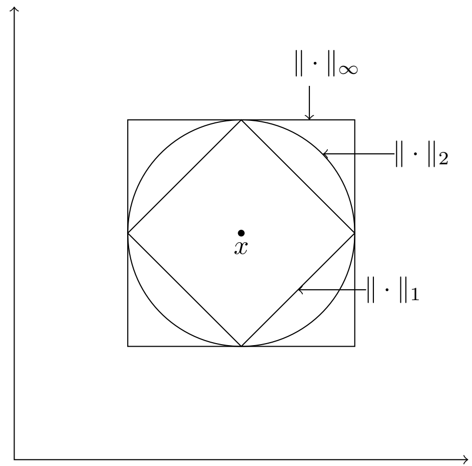
1. For x in a normed vector space $(V, \|\cdot\|)$ define the *open ball* of centre x and radius $\delta > 0$ as $B(x,\delta) = B(x,\delta,\|\cdot\|) := \{y \in V : \|y - x\| < \delta\}$.

2. $U \subset V$ is called an *open* subset if $\forall x \in U$ $\exists \delta = \delta_x > 0$ s.t. $B(x, \delta_x, \|\cdot\|) \subset U$

Example:

- In $(\mathbb{R}, |\cdot|)$, $B(x,\delta) = (x - \delta, x + \delta)$.
- In $\mathbb{R}^2$, see picture.

- In $(\mathcal{C}[0,1], \|\cdot\|_\infty)$, $B(f, \delta, \|\cdot\|_\infty) = \{g : [0,1] \to \mathbb{R} \text{ cts} : f - \delta < g < f + g\}$



Lemma 42: $B(x, \delta)$ is an open subset of $(V, \|\cdot\|)$ and $B(x, \delta) = x + \delta B(0, 1)$

Proof. Let $y \in B(x, \delta)$. Then $\|y - x\|_V < \delta \implies \delta - \|y - x\|_V > 0$. Then $\|z - y\|_V < \delta - \|y - x\|_V \implies \|z - x\|_V \leq \|z - y\|_V + \|y - x\|_V < \delta$ so $B(y, \delta - \|y - x\|_V \subset B(x, \delta)$
$\|v\|_V < 1 \Leftrightarrow \|\delta v\|_V < \delta \Leftrightarrow \|(x + \delta v) - x\|_V < \delta \Leftrightarrow x + \delta V \in B(x, \delta)$ □

Lemma 43: If $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms on V then U open in $(V, \|\cdot\|_b) \Leftrightarrow U$ is open in $(V, \|\cdot\|_a)$.

Proof. ($\implies$): Take $K \in \mathbb{R}_+$ s.t. $\|v\|_b \leq K\|v\|_a$ $\forall v \in V$. Then $\|v\|_a < \frac{\delta}{K} \implies \|v\|_b < \delta$ so $B(x, \frac{\delta}{K}, \|\cdot\|_a) := \{y \in V : \|y - x\|_a < \frac{\delta}{K}\} \subset \{y \in V : \|y - x\|_b < \delta\}$. If U is open in $(V, \|\cdot\|_b)$ then $\forall x \in U$ $\exists \delta > 0$ s.t. $B(x, \delta, \|\cdot\|_b) \subset U$ and then $B(x, \frac{\delta}{K}, \|\cdot\|_a) \subset U$ so U is open in $(V, \|\cdot\|_a)$.
($\implies$): identical proof. □

Example: $B(0, 1, \|\cdot\|_2)$ is open in $(\mathbb{R}^2, \|\cdot\|_2)$ but also in $(\mathbb{R}^2, \|\cdot\|_\infty)$, $\|v\|_2 \leq \sqrt{2}\|v\|_\infty$

Remark: The definition of continuity for $f : (V, \|\cdot\|_V) \to (W, \|\cdot\|_W)$ can be defined in terms of open balls.

Definition 14 (equivalently): f is continuous at $x \in V$ if $\forall \varepsilon > 0$ $\exists \delta_x(\varepsilon) > 0$ s.t. $f(B(x, \delta_x, \|\cdot\|_V)) \subset B(f(x), \varepsilon, \|\cdot\|_W)$

Proposition 44: A function $f : V \to W$ is $(\|\cdot\|_V, \|\cdot\|_W)$-continuous $\Leftrightarrow \forall U$ open in $(W, \|\cdot\|_W)$, $f^{-1}(U)$ is open in $(V, \|\cdot\|_V)$.

Proof. ($\implies$): Let U be open in $(W, \|\cdot\|_W)$ and $x \in f^{-1}(U)$. Then $f(x) \in U$ and $\exists \varepsilon_{f(x)} > 0$ s.t. $B(f(x), \varepsilon_{f(x)}, \|\cdot\|_W) \subset U$. Since f is continuous at x $\exists \delta_x > 0$ s.t. $f(B(x, \delta_x, \|\cdot\|_V)) \subset B(f(x), \varepsilon_{f(x)}, \|\cdot\|_W)$. Hence $f(B(x, \delta_x, \|\cdot\|_V) \subset U$ so $B(x, \delta_x, \|\cdot\|_V) \subset f^{-1}(U)$. Therefore $f^{-1}(U)$ is open in $(V, \|\cdot\|_V)$.
($\implies$): Given $x \in V$ and $\varepsilon > 0$, $B(f(x), \varepsilon, \|\cdot\|_W)$ is open in $(W, \|\cdot\|_W)$ (by Lemma 42) so by assumption $f^{-1}(B(f(x), \varepsilon, \|\cdot\|_W))$ is open in $(V, \|\cdot\|_V)$. Also $x \in f^{-1}(B(f(x), \varepsilon, \|\cdot\|_W))$ since $f(x) \in B(f(x), \varepsilon, \|\cdot\|_W)$ so $\exists \delta_x > 0$ s.t. $B(x, \delta_x, \|\cdot\|_W) \subset f^{-1}(B(f(x), \varepsilon, \|\cdot\|_W))$.
Thus $f(B(x, \delta, \|\cdot\|_V)) \subset B(f(x), \varepsilon, \|\cdot\|_W)$. So f is $(\|\cdot\|_V, \|\cdot\|_W)$-continuous. □

Definition 16: $U \subset V$ is called a *closed* subset if $V \setminus U$ is open.

Example: in $(\mathbb{R}, |\cdot|)$, $[a, b]$ is closed because $\mathbb{R} \setminus [a, b] = (\infty, a) \cup (b, \infty)$. $[a, b)$ is neither open ($\forall \delta > 0$ $\exists$ no $B(a, \delta) \in [a, b)$) nor closed ($[b, \infty)$ is not open).

Proposition 45: Let $(V, \|\cdot\|)$ be a normed vector space and $U \subset V$. Then the following are equivalent:

1. U is closed.

2. If $x_n \in U$ $\forall n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ then $x \in U$.

Proof.

$1 \Longrightarrow 2$: Let $x_n \to x$ as $n \to \infty$ and $x_n \in U$ $\forall n$. Suppose $x \notin U$. Then $x \in V \setminus U$. Since $V \setminus U$ is open $\exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subset V \setminus U$. Now $\|x_n - x\| \to 0$ as $n \to \infty$ so $\exists m \in \mathbb{N}$ s.t. $\|x_m - x\| < \varepsilon$ and hence $x_m \in B(x, \varepsilon)$ so $x_m \in V \setminus U$ – contradiction.

$2 \Longrightarrow 1$: suppose U is *not* closed. Then $V \setminus U$ is not open so $\exists x \in V \setminus U$ s.t. no $B(x, \varepsilon)$ is contained in $V \setminus U$. Choosing $\varepsilon = \frac{1}{n}$, $n = 1, 2, ...$ get a sequence of points (x_n) in V s.t. $\forall n$, $x_n \in B(x, \frac{1}{n})$ and $x_n \in U$. Thus $\|x_n - x\| \to 0$ as $n \to \infty$ and by assumption $x \in U$ – contradiction.

□

The set $\{y \in V : \|y - x\|_V \ \leq \delta\} =: \bar{B}(x, \delta, \| \cdot \|_V)$ is called a *closed ball* (of centre x, radius δ). It is a closed set.

Proof. If $y_n \in \bar{B}(x, \delta)$ $\forall n$ and $y_n \to y$ as $n \to \infty$, then $\lim_{n\to\infty} \|y_n - x\| \leq \delta \Leftrightarrow \|y - x\| \leq \delta$ □

In conclusion, U is open in V when U "has no boundary points", i.e. from any point in U can go some positive distance in V without going outside of U. U is closed iff it contains its limit points.

12 Contraction Mapping and Solution to an ODE

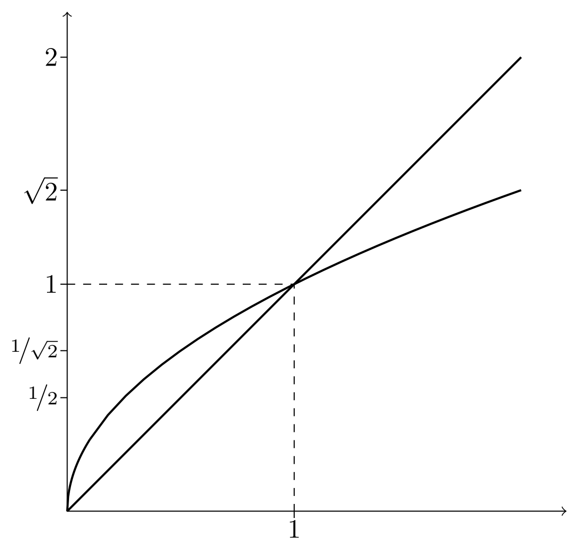
Theorem 46 (Contraction Mapping Theorem/Banach fixed point Theorem/Method of Successive Iterations): Let $(V, \|\cdot\|)$ be a Banach space, $U \subset V$ a non-empty closed subset. $0 < K < 1$, $f : U \to U$ a function satisfying:

$$x, y \in U \implies \|f(x) - f(y)\| \leq K\|x - y\|$$

(Such an f is called a *contraction mapping*) Then f has a unique *fixed point* $z \in U$, i.e. a point z s.t. $f(z) = z$. Moreover, $\forall x \in U$, $(x_n) \to z$ as $n \to \infty$, where we define inductively $x_{n+1} = f(x_n)$ with $x_0 = x$.

Example: $(V, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $U = [\frac{1}{2}, 2]$, $f : x \mapsto \sqrt{x}$
f maps U into itself: $[\frac{1}{\sqrt{2}}, \sqrt{2}] \subset [\frac{1}{2}, 2]$. $\sqrt{x} = x$ in $[\frac{1}{2}, 2] \Leftrightarrow x = 1$ is a unique fixed point. $\Gamma|_{[\frac{1}{2},2]}$ is a contraction with $K = \frac{1}{\sqrt{2}} < 1$ by MVT.

$$|\sqrt{x} - \sqrt{y}| = \left|\frac{1}{2}\sqrt{\xi}\right| |x - y| \leq \frac{1}{\sqrt{2}}|x - y|, \; x \leq \xi \leq y \text{ (WLOG)}$$



Proof of Theorem 46.
Existence: pick $x_0 \in U$ and let $x_n = f^n(x_0)$, where $f^n(z) := f \circ f \circ ... \circ f(z)$ (the nth composition of f). For $n \geq 1$:

$$\begin{aligned} \|x_n - x_{n+1}\| &= \|f(x_{n-1}) - f(x_n)\| \\ &\leq K\|x_{n-1} - x_n\| \\ &\leq K^2\|x_{n-2} - x_{n-1}\| \\ &\;\;\vdots \\ &\leq K^n\|x_0 - x_1\| && \text{(by induction)} \end{aligned}$$

If $m > n$ the triangle inequality gives:

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + ... + \|x_{m-1} - x_m\| \\ &\leq (K^n + K^{n+1} + ... + K^{m-1})\|x_0 - x_1\| \\ &\leq \frac{K^n}{1 - K} \cdot \|x_0 - x_1\| \to 0 \text{ as } n \to \infty \end{aligned}$$

Therefore (x_n) is Cauchy so converges to some point $\bar{x} \in U$ since U is closed (by Proposition 45).

To show $\bar{x}$ is a fixed point:

1. If $\|f(\bar{x}) - \bar{x}\| > 0$ let $\varepsilon = \frac{1}{3}\|f(\bar{x}) - \bar{x}\|$. $\exists N \in \mathbb{N}$ s.t. $\|x_n - \bar{x}\| < \varepsilon \; \forall n \geq N$, then $\|f(x_n) - f(\bar{x})\| \leq K\|x_n - \bar{x}\| < K\varepsilon < \varepsilon$.
   So $\|\bar{x} - f(\bar{x})\| \leq \|\bar{x} - x_{n+1}\| + \|x_{n+1} - f(\bar{x})\| < \varepsilon + \varepsilon = \frac{2}{3}\|f(\bar{x}) - \bar{x}\|$
   $\therefore \|\bar{x} - f(\bar{x})\| = 0$, $\therefore f(\bar{x}) = \bar{x}$
2. Alternatively: f is a contraction $\implies$ f is continuous (proof: $\|f(x) - f(y)\| < K\|x - y\|$: take $\delta = \frac{\varepsilon}{2K}$). Then $f(\bar{x}) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \bar{x}$.

Uniqueness: If $f(z) = z$ and $f(y) = y$ then $\|z - y\| = \|f(z) - f(y)\| \leq K\|z - y\|$ $(K < 1)$. $\therefore \|z - y\| = 0$, $\therefore z = y$ so f has at most one fixed point in U. ☐

Lemma 47: If $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $x_0, y_0 \in \mathbb{R}$ and $\delta > 0$ then the following are equivalent:

1. $y : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ is differentiable and satisfies $\begin{cases} \frac{dy}{dt} = F(x, y), \\ y(x_0) = y_0 \end{cases} \quad \forall x \in [x_0 - \delta, x_0 + \delta]$
2. $y : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ is continuous and satisfies $y(x) = y_0 + \int_{x_0}^{x} F(s, y(s))\, ds \; \forall\, x \in [x_0 - \delta, x_0 + \delta]$

Proof.

$1 \implies 2$: y differentiable $\implies$ y continuous and since F is continuous, FTC 2 (Theorem 18) $\implies$ $y(x) - y(x_0) = \int_{x_0}^{x} F(s, y(s))\, ds$. Since $y(x_0) = y_0$, $y(x) = y_0 + \int_{x_0}^{x} F(s, y(s))\, ds \; \forall\, x \in [x_0 - \delta, x_0 + \delta]$.

$2 \implies 1$: immediate consequence of FTC.

☐

Picard iteration method: let $y^{(0)}(x) = y_0$, $y^{(n+1)}(x) = y_0 + \int_{x_0}^{x} F(s, y^{(n)}(s))\, ds$, $n = 0, 1, 2, ...$ ($y^{(n)} = n$th approximation to solution)

Example:

$$\begin{cases} \frac{dy}{dx} = y, \\ y(0) = y_0 \end{cases} \qquad y^{(1)}(x) = y_0 + \int_0^x y_0\, ds = y_0(1 + x)$$

$$y^{(2)}(x) = y_0 + \int_0^x y_0(1 + s)\, ds = y_0\left(1 + x + \frac{x^2}{2}\right)$$

$$\vdots$$

$$y^{(n)}(x) = y_0 + \int_0^x y^{(n-1)}(s)\, ds = y_0 + \int_0^x y_0\left(1 + s + ... + \frac{s^{n-1}}{(n-1)!}\right) ds = y_0\left(1 + x + ... + \frac{x^n}{n!}\right)$$

(by induction)

$$\to y_0 e^x \text{ as } n \to \infty$$

Theorem 48: Let $F : R := [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \to \mathbb{R}$ be continuous and assume $\exists L > 0$ s.t.

$$|F(x, y) - F(x, z)| \leq L|y - z| \qquad (*)$$

$\forall (x, y), (x, z) \in R$. Then $\exists \delta \in (0, a]$ and a unique $\mathcal{C}^1$ function $h : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ with $h(x_0) = y_0$ and satisfying $\frac{dh}{dx} = F(x, h(x)) \; \forall x \in [x_0 - \delta, x_0 + \delta]$.

Note: $(*)$ is a Lipschitz condition on the second coordinate. If $\frac{dF}{dy}(x, y)$ exists and is continuous then $(*)$ follows from the MVT: $F(x, y) - F(x, z) = \frac{dF}{dy}(x, \xi)(y - z)$ so take $L = \max_{(x,y)\in R} \left|\frac{dF}{dy}(x, y)\right|$.

Proof. Let $M := \max_{(x,y)\in R} |F(x, y)|$. Take $\delta > 0$ s.t. $\begin{cases} \delta L < 1 \\ \delta M \leq b \end{cases}$

Let $V = (\mathcal{C}[x_0 - \delta, x_0 + \delta], \|\cdot\|_\infty)$, let $U := \{f \in \mathcal{C}[x_0 - \delta, x_0 + \delta] : f(x_0) = y_0 \text{ and } \|f - y_0\|_\infty \leq b\}$.

Then U is a non-empty closed subset of Banach space V so U is complete. Define the Picard operator $P : U \to V : f \mapsto Pf$. $(Pf)(x) := y_0 + \int_{x_0}^{x} F(s, f(s))\, ds$. From Lemma 47, $h : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ is a solution of IVP $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \Leftrightarrow (Ph)(x) = h(x)\ \forall x \in [x_0 - \delta, x_0 + \delta]$ (i.e. h is a fixed point of the Picard operator)

To show P maps U into itself: $(Pf)(x) = y_0 + \int_{x_0}^{x} F(s, f(s))\, ds$ so $y(x_0) = y_0$ and $|(Pf)(x) - y_0| \leq |x - x_0| \sup_{x \in [x_0 - \delta, x_0 + \delta]} |F(x, f(x))|$ hence $\|Pf - y_0\|_\infty \leq \delta M$ and thus $Pf \in U$ if $\delta M \leq b$.

To show P is a contraction:

$$\begin{aligned} \|Pf - Pg\|_\infty &= \sup_{x \in [x_0 - \delta, x_0 + \delta]} \left| \int_{x_0}^{x} [F(s, f(s)) - F(s, g(s))]\, ds \right| \leq \delta L \sup_{x \in [x_0 - \delta, x_0 + \delta]} |f(x) - g(x)| \\ &= \delta L \|f - g\|_\infty \end{aligned}$$

So P is a contraction if $K = \delta L < 1$. The Contraction Mapping Theorem implies P has a unique fixed point, $h : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$, h has the required properties by Lemma 47. □

13 Power Series

Apply theory in chapter 9 to power series $\sum_{k=0}^{\infty} a_k x^k$, $a_k \in \mathbb{R}$

Theorem 49: Suppose $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = x_0 \neq 0$ and $0 < b < |x_0|$. Then $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[-b, b]$ and the derived series $\sum_{k=0}^{\infty} k a_k x^{k-1}$ converges uniformly on $[-b, b]$.

Proof. (Idea: use M-test and $\frac{b}{|x_0|} < 1$)
$\sum a_k x_0^k$ converges so $a_k x_0^k \to 0$ as $k \to \infty$ and all terms are bounded. Choose $k \in \mathbb{R}_+$ s.t. $|a_k x_0^k| \leq k$. If $|x| \leq b$ then:

$$|a_k x^k| \leq |a_k b^k| \leq k \left(\frac{b}{|x_0|}\right)^k =: M_k$$

$\sum M_k$ converges with $\frac{k}{1-\frac{b}{|x_0|}}$ so M-test implies $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[-b, b]$. $\sum kk \left(\frac{b}{|x_0|}\right)^{k-1}$ converges (by ratio test). Therefore, $\sum_{k=0}^{\infty} k a_k x^{k-1}$ converges uniformly on $[-b, b]$. □

Remark:

1. The same proof works for $a_k, x, x_0 \in \mathbb{C}$ and $0 < b < |x_0|$. Both series converge uniformly on $\{x \in \mathbb{C} : |x| \leq b\}$.

2. $R := \sup\{|x_0| : \sum a_k x_0^k \text{ converges}\} \in [0, \infty]$ defines the *radius of convergence* of the series and satisfies:

$$\begin{cases} z \in \mathbb{C} : |z| < R \implies \sum_k a_k x^k \text{ converges.} \\ z \in \mathbb{C} : |z| > R \implies \{a_k z^k : k \in \mathbb{N}\} \text{ is not bounded and } \sum_k a_k z^k \text{ diverges.} \end{cases}$$

Theorem 50 (termwise differentiation and integration): Let $R > 0$ and let $\sum_{k=1} a_k x^k$ be a real power series that converges pointwise on $(-R, R) \subset \mathbb{R}$ to $f : (-R, R) \to \mathbb{R}$. Then f is continuous and differentiable with $\forall x \in (-R, R)$, $f'(x) = \sum_{k=1} k a_k x^{k-1}$. If $-R < c < d < R$. Then $f|_{[c,d]}$ is regulated and $\int_c^d f = \sum_{k=0}^{\infty} a_k \frac{(d^{k+1} - c^{k+1})}{k+1}$.

Proof. Take $0 < b < x_0 < R$. Then $\sum a_k x_0^k$ converges so by Theorem 49, $\sum a_k x^k$ converges uniformly on $[-b, b]$. By Theorem 25', $f|_{[-b,b]}$ is continuous at each point of $[-b, b]$. $(-R, R) = \bigcup_{0<b<R} (-b, b)$ so $f : (-R, R)$ is continuous. Also $\sum k a_k x^{k-1}$ converges uniformly on $[-b, b]$ by Theorem 49, and by Theorem 29', f is $\mathcal{C}^1$ on $[-b, b]$ with $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$, so f is $\mathcal{C}^1$ on $(-R, R)$. The integral result follows from Theorem 24' and:

$$\int_c^d \left(\sum_{k=0}^{n} a_k x^k\right) dx = \sum_{k=0}^{n} a_k \int_c^d x^k \, dx = \sum_{k=0}^{n} a_k \frac{d^{k+1} - c^{k+1}}{k+1}$$

□

Remark: By applying Theorem 50 repeatedly we find the function $f : (-R, R) \to \mathbb{R}$ is infinitely differentiable with $f^{(k)}(0) = k! a_k$. Hence the Taylor series of f *is* $\sum_{k=0}^{\infty} a_k x^k$ which is f! *If* f is given by a power series then its Taylor series is that same series, which *does* converge to f. Hence $\begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ which has Taylor series $\sum_k 0 x^k = 0$ is *not* given by a power series.

Definition 17: Define the following functions: $\mathbb{R} \to \mathbb{R}$ (or indeed $\mathbb{C} \to \mathbb{C}$) by power series:

$$\sin: \ \sin(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \sinh: \ \sinh(x) := \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos: \ \cos(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \cosh: \ \cosh(x) := \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$\exp: \ \exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

but $\log : (0, \infty) \to \mathbb{R}$, $\log(x) := \int_1^x \frac{1}{t}\, dt$.

Proposition 51: The radius of convergence of each of the series for $\sin, \sinh, \cos, \cosh, \exp$ is ∞.

Proof. For cos, $\forall x \in \mathbb{R}$:

$$\left| \frac{(-1)^{k+1} x^{2k+2}}{(2k+2)!} \Big/ \frac{(-1)^k x^{2k}}{(2k)!} \right| = \left| \frac{x^2}{(2k+2)(2k+1)} \right| \to 0 \text{ as } k \to \infty$$

so $\forall x$ converges by the ratio test (similar method for the other functions). □

Corollary 52: On $\mathbb{R}$, $\exp' = \exp, \ \cosh' = \sinh, \ \sinh' = \cosh, \ \cos' = -\sin, \ \sin' = \cos$.

Note: also true on $\mathbb{C}$.

Proof. By Theorem 50, within $(-R, R) = \mathbb{R}$:

$$\exp'(x) = \sum_{k=1}^{\infty} \frac{k x^{k-1}}{k!} = \sum_{j=0}^{\infty} \frac{x^j}{j!} = \exp(x)$$

$$\cos'(x) = \sum_{k=1}^{\infty} \frac{(1)^k (2k) x^{2k-1}}{(2k)!} = -\sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} = -\sin(x) \qquad (\text{put } j = k-1)$$

Others similarly. □

Recall: a series $\sum a_k$ is *absolutely convergent* if $\sum |a_k| < \infty$, in which case any rearrangement of $\sum a_k$ is absolutely convergent and gives the same sum, and:

$$\left(\sum_j a_j \right) \left(\sum_k b_k \right) = \sum_{j,k} a_j b_k$$

Proposition 53: $\forall x \in \mathbb{C}$:

$$\begin{aligned}
\cosh(x) &= \frac{1}{2}(\exp(x) + \exp(-x)) \\
\sinh(ix) &= i \sin(x) \\
\sinh(x) &= \frac{1}{2}(\exp(x) - \exp(-x)) \\
\cosh(ix) &= \cos(x) \\
\exp(ix) &= \cos(x) + i \sin(x) \\
\exp(x+y) &= \exp(x) \exp(y) \\
\cos(x+y) &= \cos(x) \cos(y) - \sin(x) \sin(y) \\
\sin(x+y) &= \sin(x) \cos(y) + \sin(y) \cos(x) \\
(\cos(x))^2 + (\sin(x))^2 &= 1
\end{aligned}$$

Proof. Rearranging:

$$\frac{1}{2}(\exp(x)-\exp(-x)) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{x^k}{k!} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!} = \frac{1}{2}\sum_{k=0}^{\infty}(1-(-1)^k)\frac{x^k}{k!} = \sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!} = \sinh(x)$$

$$\exp(x+y) = \sum_{k=0}^{\infty}\frac{(x+y)^k}{k!} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{k!x^j y^{k-j}}{j!(k-j)!k!} = \left(\sum_{l=0}^{\infty}\frac{y^l}{l!}\right)\left(\sum_{j=0}^{\infty}\frac{x^j}{j!}\right) = \exp(y)\exp(x)$$

(binomial theorem)

Others, similarly:

$$1 = \cos(0) = \cos(x-x) = \cos(x)\cos(x) + \sin(x)\sin(x)$$

(using $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$) □

Proposition 52: There is $\pi > 0$ with $\forall x \in \mathbb{R}$, $\sin(x+\pi) = -\sin(x)$ and $\cos(x+\pi) = -\cos(x)$.

Proof. $\sin'(0) = \cos(0) = 1$ and $\sin(0) = 0$ so for all small positive x, $\sin(x) > 0$. For $0 < x < 12$:

$$\sin(x) < x - \frac{x^3}{3!} + \frac{x^3}{3!} - \frac{x^7}{7!} + \frac{x^9}{9!} \qquad \left(\text{since } -\tfrac{x^{11}}{11!} + \tfrac{x^{13}}{13!} < 0, \text{ and the same for the next terms}\right)$$

and $\sin(4) < 4 - \frac{4^3}{3!} + \frac{4^5}{5!} - \frac{4^7}{7!} + \frac{4^9}{9!} < -0.6617 < 0$. By the IVT (Analysis II) sin takes value 0 in $(0,4)$. Define π as the smallest x in $(0,4)$ with $\sin(x) = 0$. Then $\cos' = -\sin$ is negative on $(0,\pi)$ so the MVT implies that $\cos|_{[0,\pi]}$ is strictly decreasing. By Proposition 51 $\cos^2(\pi) + \sin^2(\pi) = 1$ so $\cos(\pi) = -1$. Hence:

$$\sin(x+\pi) = \sin(x)\cos(\pi) + \cos(x)\sin(\pi) = -\sin(x)$$
$$\cos(x+\pi) = \cos(x)\cos(\pi) - \sin(x)\sin(\pi) = -\cos(x)$$

□

Note: $\forall x \in \mathbb{R} \begin{cases} \sin(x+2\pi) = -\sin(x+\pi) = \sin(x) \\ \cos(x+2\pi) = -\cos(x+\pi) = \cos(x) \end{cases}$
i.e. these functions are periodic with period 2π and so is any linear combination of them of $\sin(kx)$ and $\cos(kx)$, and any pointwise convergent $\sum_{k\in\mathbb{N}}(a_k\cos(kx) + b_k\sin(kx))$ (recall Fourier series).