MA359 Measure Theory

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Introduction

These lecture notes are a projection of the MA359 Measure Theory course 2013/2014, delivered by Dr José Rodrigo at the University of Warwick. These notes should be virtually complete, but the tedious treasure hunt of errors will always be an open game. And, obviously, completeness and accuracy cannot be guaranteed. If you spot an error, or want the source code to fiddle with in your way, send an e-mail to `me@tomred.org`. We hope these are helpful, and good luck!

Tom and Usman ♡

Useful links

1. The up-to-date version of these notes should be found here:
   `https://www.dropbox.com/sh/zqreyxd1dyazpes/OlbaDh95ze/Year%203/MA359%20Measure%20Theory`
2. Failing that:
   `http://www.tomred.org/lecture-notes.html`
3. Students taking this course should also take a look at Lewis Woodgate's Skydrive notes:
   `https://skydrive.live.com/view.aspx?resid=AC6AC9E3BEE89219!308&app=OneNote&authkey=!AB4KXPRDOKG9QKc`
4. ...and Alex Wendland's Dropbox notes:
   `https://www.dropbox.com/sh/5m63moxv6csy8tn/LY3576RtRQ/Year%203/Measure%20theory`

We want to measure *every subset* of $\mathbb{R}$. i.e. we want a map:

$$m : \underbrace{\mathcal{P}(\mathbb{R})}_{\text{parts of } \mathbb{R}} \to [0, \infty]$$

where $m(\text{interval } (a,b)) = b - a$ (same with intervals such as $[a,b)$). e.g. $m((1,2)) = 2 - 1$.
Wish list form:

1. $$m((a,b)) = b - a$$

2. $$m(A) = m(A + h) \qquad (\forall A \subseteq \mathbb{R},\ \forall h \in \mathbb{R})$$

3. $$A = \bigcupdot_{n=1}^{\infty} A_n \implies m(A) = \sum_{n=1}^{\infty} m(A_n)$$

Claim. There isn't such an m.
Goal: "Construct" a subset $\mathbb{R}$, such that it is impossible to assign a measure and satisfy the proposition in the wish list.

1 Real Line

Agree on the measure of intervals:

$$\begin{aligned} I &= (a,b) \\ &= [a,b) \\ &= (a,b] \\ &= [a,b] \end{aligned}$$

$$m(I) = \text{"usual length" of } I = b - a$$

Definition 1. *Let $A \subseteq \mathbb{R}$:*

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} |I_k| : I_k \text{ are open intervals and } A \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

where $|I_k|$ is the length of the interval I_k. m^* is the **outer measure***.*

Construction:

1. Cover A by lots of open intervals.
2. Summing the lengths creates a set in $[0, \infty]$.
3. Compute the infimum (the existence of which is trivial, as the set in question is bounded below).

Proposition 1 (Properties of m^*)**.**

1. *$0 \leq m^*(A)\ \forall A \subseteq \mathbb{R}$.*
2. *$m^*(\mathbb{Q}) = 0$ (surprising because $\mathbb{Q}$ is dense).*
3. *m^* is defined for $\mathcal{P}(\mathbb{R})$ (it is defined for every subset of $\mathbb{R}$).*
4. *$m^*(A) \leq m^*(B)$ whenever $A \subset B$.*

5. *$m^*(I) = |I|$ for any interval I.*

6. *$m^*(A+h) = m^*(A)$ $\forall h \in \mathbb{R}$, $A \subset \mathbb{R}$.*

Proof.

1. For any interval I, $|I| \geq 0$, and as $m^*(A)$ is a greatest lower bound, $m^*(A) \geq 0$ $\forall A$.

2. Take $\{x_n\}$ to be an enumeration of $\mathbb{Q}$. Define:

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right)$$

for any fixed $\varepsilon > 0$. Notice $|I_n| = \frac{\varepsilon}{2^n}$ and $\mathbb{Q} \subset (\bigcup_{n=1}^{\infty} I_n)$. Since:

$$m^*(\mathbb{Q}) = \inf\left\{\sum_{n=1}^{\infty} |J_n| : J_n \text{ open and } \mathbb{Q} \subset \bigcup_{n=1}^{\infty} J_n\right\}$$

we have:

$$m^*(\mathbb{Q}) \leq \sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

So $0 \leq m^*(\mathbb{Q}) \leq \varepsilon$ for any $\varepsilon \geq 0$. By sending $\varepsilon \to 0 \implies m^*(\mathbb{Q}) = 0$

3. $\sum_{k=1}^{\infty} |I_k|$ is defined as it is a limit of an increasing sequence, so our infimum will always be defined.

4. Every open cover of B by intervals also covers $A \implies$ the collection of elements over which we compute $\implies m^*(A) \leq m^*(B)$.

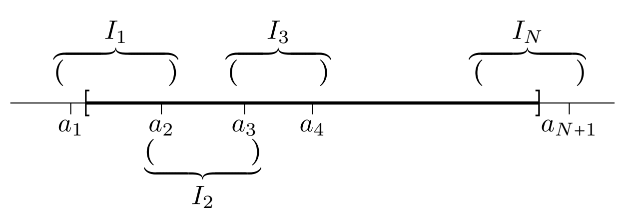
5. We will show two inequalities:

$$m^*(I) \leq |I|$$
$$m^*(I) \geq |I|$$

Now for $|I| \leq m^*(I)$:
Take a cover of I, $\{I_n\}_{n=1}^{\infty}$. Since $I = [a,b]$ there exists a finite subcover of I (as it's compact and closed). Upon relabelling the sets, say:

$$I_1, I_2, ..., I_N$$

We have our finite subcover of I:



As these are all open, there is overlap inbetween the open intervals. Choose a point in each of the overlapping intervals. i.e. choose a_j from each $I_j \cap I_{j+1}$. So $(a_1, a_2) \subset I_1$, $(a_2, a_3) \subset I_2$ and $(a_j, a_{j+1}) \subset I_j$. Now:

$$|b-a| \leq |a_{N+1} - a_1| = a_{N+1} - a_N + a_N - a_{N-1} + a_{N-1} - ... + a_2 - a_1 = \sum_{j=1}^{N} a_{j+1} - a_j$$

As $a_{j+1} - a_j \leq |I_j|$. Then:

$$|I| = |b-a| \leq \sum_{j=1}^{N} a_{j+1} - a_j \leq \sum_{j=1}^{N} |I_j| \leq \sum_{j=1}^{\infty} |I_j| \qquad \text{(for all open covers)}$$

Which implies:

$$|I| \leq \inf \left\{ \sum_{j=1}^{\infty} |I_j| : I_j \text{ open...} \right\} = m^*(I)$$

Now for $|I| \geq m^*(I)$:
Say $I = [a, b]$. Define $I_1 = (a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$, $I_j = \varnothing \ \forall j \geq 2$.

$\implies [a, b] \subset \bigcup_{j=1}^{\infty} I_j$

$\implies m^*(I) \leq |I_1| \leq b - a + \varepsilon = |I| + \varepsilon \ \forall \varepsilon > 0$

$\implies$ as $\varepsilon \to 0$, $m^*(I) \leq |I|$.

6. Reason is $|I + h| = |I|$.

□

The only property that we do not have is:

$$m^*(A \cup B) = m^*(A) + m^*(B) \qquad \text{(it's false)}$$

Another observation:

$$A \subset \bigcup_{n=1}^{\infty} A_n \implies m^*(A) \leq m^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} m^*(A_n)$$

Proof. $\forall \varepsilon > 0$ there exists a countable collection of open intervals $\{I_{n,k}\}_{k=1}^{\infty}$ such that:

$$\sum_{k=1}^{\infty} |I_{n,k}| \leq m^*(A_n) + \frac{\varepsilon}{2^n} \qquad (*)$$

If $\sum_{n=1}^{\infty} m^*(A_n) = \infty$, there is nothing to prove. Else, sum $(*)$ w.r.t. n:

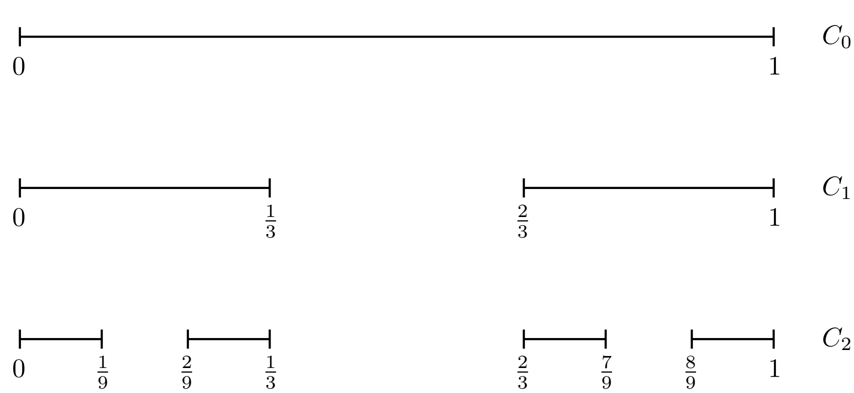
$$\sum_{n,k=1}^{\infty} |I_{n,k}| \leq \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon$$

Want to show:

$$m^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n,k}^{\infty} |I_{n,k}| \leq \left(\sum_{n=1}^{\infty} m^*(A_n) \right) + \varepsilon \qquad (**)$$

This is true $\forall \varepsilon > 0$ so send $\varepsilon \to 0$. □

1.1 Cantor set



$C := \bigcap_{n=1}^{\infty} C_n$ but there exists a bijection between C and $\mathbb{R}$.

$$m^*(C) \le m^*(C_n) \le 2^n \left(\frac{1}{3}\right)^n \to 0 \qquad \text{(as } n \to \infty\text{)}$$

So $m^*(C) = 0$. Thus measure and cardinality do not mix well...

Definition 2. *We say that $A \subset \mathbb{R}$ is* **measurable** *iff:*

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c) \qquad (\forall E \subset \mathbb{R})$$

Remark 1. *It is enough to show $m^*(E) \ge m^*(E \cap A) + m^*(E \cap A^c)$ $\forall E \subset \mathbb{R}$. This is because:*

$$E \subset (E \cap A) \cup (E \cap A^c) \implies m^*(E) \le m^*((E \cap A) \cup (E \cap A^c)) \le m^*(E \cap A) + m^*(E \cap A^c)$$

by the above proposition.

Example 1 (Examples of measurable sets)**.**

- *$\mathbb{Q}$ is measurable, as $m^*(\mathbb{Q}) = 0$.*
- *Any set $A \subset \mathbb{R}$ with $m^*(A) = 0$.*

  Proof.

$$\begin{aligned} (E \cap A) \subset A &\implies m^*(E \cap A) \le m^*(A) = 0 \\ (E \cap A^c) \subset E &\implies m^*(E \cap A^c) \le m^*(E) \\ &\implies m^*(E) \ge m^*(E \cap A) + m^*(E \cap A^c) \end{aligned}$$

□

- *$\mathbb{R} \setminus \mathbb{Q}$ is by the lemma below.*

Lemma 1. *A measurable $\implies$ A^c measurable.*

Proof.

$$m^*(E) = m^*(E \cap A^c) + m^*(E \cap (A^c)^c)$$

□

Proposition 2. *Intervals are measurable.*

Proof. Want to show:

$$m^*(E) = m^*(E \cap I) + m^*(E \cap I^c) \qquad (\forall E \subset \mathbb{R})$$

First, take an open cover of E by intervals, say $\{E_k\}_{k=1}^{\infty}$. $I \cap E_k$ is an interval $\forall k$. $I^c \cap E_k$ is at most two intervals $\forall k$.
(From $\{E_k\}$ it isn't possible to construct (open) covers of $I \cap E$ and $I \cap E^c$)
$(I \cap E_k) \subset A_k$ for A_k an open interval.
$(I^c \cap E_k) \subset (B_k \cup C_k)$ for B_k, C_k open intervals.
Choose such that:

$$|A_k| + |B_k| + |C_k| < |I_k| + \frac{\varepsilon}{2^k}$$

Now, $\{I_k\}$ cover E:

$$\sum_{n=1}^{\infty} \left(|I_k| + \frac{\varepsilon}{2^k}\right) \ge \sum_{n=1}^{\infty} |A_k| + |B_k| + |C_k|$$

Also:

$$\bigcup_{k=1}^{\infty}(A\cap I_k)\subset\bigcup_{k=1}^{\infty}A_k\ \&\ \bigcup_{k=1}^{\infty}(A\cap I_k)=A\cap\left(\bigcup_{k=1}^{\infty}I_k\right)$$
$$\&\ A\cap E\subset\left(\bigcup_{k=1}^{\infty}I_k\right)$$

(Similarly for $|B_k|+|C_k|$)
So:

$$\sum_{k=1}^{\infty}|A_k|+|B_k|+|C_k|\geq m^*(E\cap A)+m^*(E\cap A^c)$$
$$\implies m^*(E\cap A)+m^*(E\cap A^c)\geq\left(\sum_{k=1}^{\infty}|I_k|\right)+\varepsilon$$

By taking the infimum over all possible covers:

$$m^*(E\cap A)+m^*(E\cap A^c)\leq m^*(E)+\varepsilon$$

Finally, let $\varepsilon\to 0$. □

Proposition 3. *A,B measurable $\implies A\cup B$ and $A\cap B$ measurable.*

Proof. We know $m^*(F)=m^*(F\cap A)+m^*(F\cap A^c)\ \forall F$. Take $F=E\cap(A\cup B)$ for some E. We want:

$$\begin{aligned}m^*(E)&=m^*(E\cap(A\cup B))+m^*(E\cap(A\cup B)^c)\\ m^*(E\cap(A\cup B))&=m^*(E\cap(A\cup B)\cap A)+m^*(E\cap(A\cup B)\cap A^c)\\ &=m^*(E\cap A)+m^*(E\cap B\cap A^c)\end{aligned}$$

Now:

$$\begin{aligned}m^*(E)&=m^*(E\cap A)+m^*(E\cap A^c) && \text{(as } A \text{ is measurable)}\\ &=m^*(E\cap A)+m^*(E\cap A^c\cap B)+m^*(E\cap A^c\cap B^c) && \text{(as } B \text{ is measurable)}\\ &=m^*(E\cap(A\cup B))+m^*(E\cap(A\cup B)^c)\end{aligned}$$

For intersection:

$$\begin{aligned}A^c \text{ and } B^c \text{ measurable} &\implies A^c\cup B^c \text{ measurable}\\ &\implies (A^c\cup B^c)^c \text{ measurable}\\ &\implies A\cap B \text{ measurable.}\end{aligned}$$

□

Proposition 4. *Let $A_1,...,A_N$ measurable and pairwise disjoint. Then:*

$$m^*\left(E\cap\dot{\bigcup_{i=1}^{N}}A_i\right)=\sum_{i=1}^{N}m^*(E\cap A_i)\qquad(\forall E)$$

Note, if $E=\mathbb{R}$, then:

$$m^*\left(\dot{\bigcup_{i=1}^{N}}A_i\right)=\sum_{i=1}^{N}m^*(A_i)$$

Proof. By induction. $N = 1$ is trivial. Assume true for $1, ..., N$. Then:

$$\begin{aligned} m^*\left(E \cap \bigcup_{n=1}^{N+1} A_i\right) &= m^*\left(\left(E \cap \bigcup_{n=1}^{N+1} A_i\right) \cap A_{N+1}\right) + m^*\left(\left(E \cap \bigcup_{n=1}^{N+1} A_i\right) \cap A_{N+1}^c\right) \\ &= m^*(E \cap A_{N+1}) + m^*\left(E \cap \bigcup_{n=1}^{N} A_i\right) \\ &= m^*(E \cap A_{N+1}) + \sum_{n=1}^{N} m^*(E \cap A_i) && \text{(by induction)} \\ &= \sum_{i=1}^{N+1} (E \cap A_i) \end{aligned}$$

□

Proposition 5. *Let* $\{A_i\}_{i=1}^{\infty}$ *be measurable. Then* $\bigcup_{i=1}^{\infty} A_i$ *is measurable. Moreover, if* A_i *are pairwise disjoint then:*

$$m^*\left(\dot{\bigcup_{i=1}^{\infty}} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i)$$

Proof. Let $B \coloneqq \bigcup_{i=1}^{\infty} A_i$ & $B_n \coloneqq \bigcup_{i=1}^{n} A_i$ which is measurable by a previous proposition. Want to show:

$$m^*(E) = m^*(E \cap B) + m^*(E \cap B^c)$$

Assume for the moment that A_i are pairwise disjoint. We know that $m^*(E) = m^*(E \cap B_n) + m^*(E \cap B_n^c)$:

$$\begin{aligned} B_n \subset B &\implies B^c \subset B_n^c \\ &\implies (E \cap B_n^c) \supset (E \cap B^c) \\ &\implies m^*(E \cap B_n^c) \geq m^*(E \cap B^c) \end{aligned}$$

Thus:

$$\begin{aligned} m^*(E) &\geq m^*(E \cap B_n) + m^*(E \cap B^c) \\ &\geq \underbrace{m^*(E \cap B_n)}_{= m^*(E \cap \bigcup_{i=1}^{n} A_i)} + m^*(E \cap B^c) \\ &\geq \sum_{i=1}^{n} m^*(E \cap A_i) + m^*(E \cap B^c) \end{aligned}$$

Now, LHS ≥ RHS and LHS is independent of n, so:

$$\begin{aligned} \text{LHS} &\geq \lim_{n \to \infty} \text{RHS} \\ m^*(E) &\geq \sum_{i=1}^{\infty} m^*(E \cap A_i) + m^*(E \cap B^c) \end{aligned}$$

Now, consider:

$$\begin{aligned} m^*(E \cap B) &= m^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) \\ &= m^*\left(\bigcup_{i=1}^{\infty} (E \cap A_i)\right) \\ &\leq \sum_{i=1}^{\infty} m^*(E \cap A_i) \end{aligned}$$

So $m^*(E) \geq m^*(E \cap B) + m^*(E \cap B^c)$. But:

$$\begin{aligned} & m^*(E) \leq m^*(E \cap B) + m^*(E \cap B^c) \\ \implies & m^*(E) = \sum_{i=1}^{\infty} m^*(E \cap A_i) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \end{aligned} \qquad (\forall E)$$

Thus, take $E = \dot\bigcup_{i=1}^{\infty} A_i$, then:

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i) + m^*(\varnothing)$$

Finally, need to show the extra hypothesis of pairwise disjoint. Define:

$$\begin{aligned} W_1 &:= A_1 \\ W_2 &:= A_2 \setminus A_1 = A_2 \cap A_1^c && \text{(measurable)} \\ W_3 &:= A_3 \setminus (A_1 \cup A_2) && \text{(measurable)} \\ &\vdots \\ W_n &:= A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right) \end{aligned}$$

Thus:

$$\bigcup_{i=1}^{\infty} A_i = \dot\bigcup_{i=1}^{\infty} W_i$$

W_i are measurable and pairwise disjoint. □

Observation: $\{B_i\}_{i=1}^{\infty}$ measurable $\implies \bigcap_{i=1}^{\infty} B_i$ measurable.

Proposition 6. *List of properties of measurable sets:*

- *Complements, countable unions and intersections of measurable sets are measurable.*
- *Intervals are measurable.*
- *(Countable additivity) A_i measurable and pairwise disjoint $\implies$ $m^*\left(\dot\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i)$.*
- *(Countinuity) A_i measurable and $A_1 \supset A_2 \supset \ldots \supset A_n \supset A_{n+1} \supset \ldots$ and B_i measurable and $B_1 \subset B_2 \subset \ldots \subset B_n \subset B_{n+1} \subset \ldots$ and $m^*(A_1) < \infty$. Then $\bigcap_{i=1}^{\infty} A_i$ & $\bigcup_{i=1}^{\infty} B_i$ are measurable. Moreover:*

$$m^*\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} m^*(A_i) \ \& \ m^*\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{i \to \infty} m^*(B_i)$$

- *(Translation invariance) A measurable $\implies$ $A + h$ measurable.*

$$m^*(A) = m^*(A + h)$$

- *Open and closed sets are measurable.*
- *(Approximation property) A measurable, then $\forall \varepsilon > 0$ $\exists B$ closed, $\exists C$ open, $B \subset A \subset C$ s.t. $m^*(C \setminus B) < \varepsilon$. Moreover, if $m^*(A) < \infty$ then B can be taken compact.*

Proof of continuity. First, we don't need $m^*(A_1) < \infty$, we need $m^*(A_n) < \infty$ for some n, as for $m^*(A_1) = \infty$, $m^*(A_2) < \infty$. We have:

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=2}^{\infty} A_i$$

as $A_1 \supset A_2$.
Let's do B_is first. $B_1 \subset B_2 \subset B_3 \subset \ldots$. Create a disjoint collection whose union is $\bigcup_{i=1}^{\infty} B_i$. Define:

$$\begin{aligned} C_1 &= B_1 \\ C_2 &= B_2 \setminus B_1 \\ C_3 &= B_3 \setminus B_2 \\ &\vdots \\ C_n &= B_n \setminus B_{n-1} \end{aligned}$$

Notice:

$$\dot{\bigcup_{i=1}^{\infty}} C_i = \bigcup_{i=1}^{\infty} B_i,$$

so:

$$\lim_{n\to\infty} m^*(B_n) = \lim_{n\to\infty} m^*\left(\bigcup_{i=1}^{n} C_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} m^*(C_i) = \sum_{i=1}^{\infty} m^*(C_i) = m^*\left(\dot{\bigcup_{i=1}^{\infty}} C_i\right) = m^*\left(\bigcup_{i=1}^{\infty} B_i\right)$$

$A_1 \supset A_2 \supset \ldots$ measurable and $m^*(A_1) < \infty$. Construct increasing set, define $D_n := A_1 \cap A_n^c$ measurable. $D_1 \subset D_2 \subset \ldots$
So we know $m^*(\bigcup_{n=1}^{\infty} D_n) = \lim_{n\to\infty} m^*(D_n)$, and:

$$A_1 = A_n \,\dot{\cup}\, D_n \implies m^*(A_1) = m^*(A_n) + m^*(D_n), \qquad (\forall n)(*)$$

and:

$$A_1 = \bigcap_{n=1}^{\infty} A_n \,\dot{\cup}\, \bigcup_{n=1}^{\infty} D_n \implies m^*(A_1) = m^*\left(\bigcap_{n=1}^{\infty} A_n\right) + m^*\left(\bigcup_{n=1}^{\infty} D_n\right).$$

So:

$$m^*(A_1) = m^*\left(\bigcap_{n=1}^{\infty} A_n\right) + \lim_{n\to\infty} m^*(D_n) \qquad (**)$$

and, by $(*)$:

$$m^*(A_1) = \lim_{n\to\infty} m^*(A_n) + \lim_{n\to\infty} m^*(D_n)$$

as $m^*(A_1) < \infty$ and $m^*(\cdot) \geq 0$. So, by $(**)$:

$$\implies m^*\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} (A_n)$$

□

Proof that open & closed sets are measurable.
Claim. Every open set in $\mathbb{R}$ can be written like:

$$U = \bigcup_{n=1}^{\infty} I_n \qquad \text{(for open intervals } I_n\text{)}$$

Thus open sets in $\mathbb{R}$ are measurable. So closed are too as complements are measurable. □

Proof of approximation property. A is measurable, want to show that $\forall \varepsilon > 0\ \exists B, C$ with $B \subset A \subset C$ s.t. $m^*(C \setminus B) < \varepsilon$. First, assume $A \subset J$, J a closed & bounded interval. Since $m^*(A) < \infty$, there exists a cover $\{I_j\}_{j=1}^{\infty}$ by open intervals:

$$A \subseteq \bigcup_{j=1}^{\infty} I_j$$

such that:

$$\sum_{j=1}^{\infty} |I_j| \leq m^*(A) + \frac{\varepsilon}{2}$$

and:

$$m^*\left(\bigcup_{j=1}^{\infty} I_j\right) \leq \sum_{j=1}^{\infty} |I_j| \leq m^*(A) + \frac{\varepsilon}{2}$$

Define:

$$C := \bigcup_{j=1}^{\infty} I_j$$

C is open and $\underbrace{m^*(C) - m^*(A)}_{= m^*(C \setminus A) \text{ as } A \subseteq C} \leq \frac{\varepsilon}{2}$.

To find B, consider $J \setminus A$ (which is measurable). We can find an open set O s.t. $(J \setminus A) \subset O$ and:

$$m^*(O) - m^*(J \setminus A) < \frac{\varepsilon}{2} \tag{†}$$

Define $B := J \setminus O = J \cap O^c$ (closed from finite intersections of closed sets). As B is measurable:

$$\begin{aligned} m^*(C) &= \underbrace{m^*(B \cap C)}_{=B} + m^*(C \cap B^c) \\ &= m^*(B) + m^*(C \setminus B) \end{aligned}$$

So:

$$\begin{aligned} m^*(C \setminus B) &= m^*(C) - m^*(B) \\ &= \underbrace{m^*(C) - m^*(A)}_{\leq \frac{\varepsilon}{2}} + m^*(A) - m^*(B) \end{aligned}$$

So all that is left is to show $m^*(A) - m^*(B) < \frac{\varepsilon}{2}$:

$$m^*(J) \leq m^*(O \cup B) \leq m^*(O) + m^*(B) < m^*(J \setminus A) + m^*(B) + \frac{\varepsilon}{2}$$

But:

$$\begin{aligned} & m^*(J) = m^*(A) + m^*(J \setminus A) \\ \implies \quad & m^*(A) + m^*(J \setminus A) < m^*(J \setminus A) + m^*(B) + \frac{\varepsilon}{2} \\ \implies \quad & m^*(A) < m^*(B) + \frac{\varepsilon}{2} \qquad \text{(as } m^*(J \setminus A) < \infty) \\ \implies \quad & m^*(A) - m^*(B) < \frac{\varepsilon}{2} \end{aligned}$$

Finally, we remove our assumption. Define:

$$\{A_n\}_{-\infty}^{\infty} := A \cap [n, n+1]$$

For each A_n fine $B_n \subset A_n \subset C_n$ with B_n closed, C_n open and $m^*(C_n \setminus B_n) \leq \frac{\varepsilon}{2^{|n|}}$. Then:

$$\bigcup B_n \subset \bigcup A_n \subset \bigcup C_n$$

So:

$$\bigcup B_n \subset A \subset \bigcup C_n$$

And let $C = \bigcup C_n$, $B = \bigcup B_n$ open.
Exercise to show that $\bigcup B_n$ is closed. Also:

$$m^*(C \setminus B) \leq m^*\left(\bigcup_{n=1}^{\infty} (C_n \setminus B_n)\right) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{|n|}} = m\varepsilon \qquad \text{(for some number } m)$$

□

2 General Measures

Let X be any non-empty set.

Definition 3. *An* **algebra** *of sets on X is a non-empty collection $\mathcal{A}$ that satisfies:*

- $Y \in \mathcal{A} \implies Y^c \in \mathcal{A}$
- $Y_1, ..., Y_n \in \mathcal{A} \implies \bigcup_{i=1}^{n} Y_i \in \mathcal{A}$

Definition 4. *A* **σ-algebra** *of sets on X is a non-empty collection of sets $\mathcal{A}$ that satisfies:*

- *$\mathcal{A}$ is closed under complements*
- *$\mathcal{A}$ is closed under countable unions*

Observations: Collection of measurable sets from Chapter 1 is a σ-algebra.

Example 2. *Let X be any infinite set. Consider:*

$$\mathcal{A} = \{E \subset X \text{ such that } E \text{ countable or } E^c \text{ countable}\}$$

Exercise: check $\mathcal{A}$ is a σ-algebra.

Observations:

- Every σ-algebra is an algebra
- If $\mathcal{A}$ is an algebra, $\varnothing \in \mathcal{A}, X \in \mathcal{A}$
- The word union can be changed with intersection (Exercise)

Proposition 7. *An algebra that is closed under countable disjoint unions is a σ-algebra.*

Proof. Given $\{A_i\}_{i=1}^{\infty}$, $A_i \in \mathcal{A}$ we want to show:

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$$

Construct out of $\{A_i\}_{i=1}^{\infty}$ a collection of pairwise disjoint collection sets B_i such that $\dot\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$:

$$\begin{aligned} B_1 &:= A_1 \\ B_2 &:= A_2 \setminus A_1 \\ B_n &:= A_n \setminus \bigcup_{i=1}^{n-1} A_i \end{aligned}$$

It is clear they are disjoint by construction

$$\begin{aligned} \bigcup_{i=1}^{n} A_i &= \dot\bigcup_{i=1}^{n} B_i \\ \bigcup_{i=1}^{\infty} A_i &= \dot\bigcup_{i=1}^{\infty} B_i \in \mathcal{A} && \text{(by assumption)} \end{aligned}$$

□

Observation: any arbitrary intersection of σ-algebras is a σ-algebra

Definition 5. *Let X be a non-empty set, $\mathcal{M}$ a σ-algebra of X. A function $\mu : \mathcal{M} \to [0, \infty]$ is a* **measure** *if it satisfies:*

- $\mu(\varnothing) = 0$

- *Countable additivity i.e* $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$*,* E_i *pairwise disjoint. Then:*

$$\mu\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Definition 6. *Any* $\mu : \mathcal{M} \to [0, \infty]$ *such that*

- $\mu(\varnothing) = 0$
- *if* $\{E_i\}_{i=1}^{N} \subset \mathcal{M}$*,* E_i *pairwise disjoint, then:*

$$\mu\left(\dot{\bigcup_{i=1}^{N}} E_i\right) = \sum_{i=1}^{N} \mu(E_i)$$

is called a **finitely additive measure** *(not necessarily a measure).*

Notation:

- A pair $(X, \mathcal{M})$, where X is a non-empty set, $\mathcal{M}$ is a σ-algebra, is called a **measurable space**.
- A triplet $(X, \mathcal{M}, \mu)$ is a **measure space**.
- Given $(X, \mathcal{M}, \mu)$, if $\mu(X) < \infty$ then μ is a **finite measure**.
- If $\mu(X) = \infty$ but there is a collection of sets $\{E_i\}_{i=1}^{\infty}$ such that $E_i \in \mathcal{M}$, $\bigcup_{i=1}^{\infty} E_i = X$ and $\mu(E_i) < \infty$, then μ is **σ-finite**.

Example 3. *Let* $X = \mathbb{R}$*,* $\mathcal{M}$ *be the collection of measurable sets from Chapter 1,* $\mu = m^*$*. Then* $\mu(X) = m^*(\mathbb{R}) = \infty$ *but if* $E_n = [-n, n]$*, then* $\bigcup_{n=1}^{\infty} E_n = X$ *and* $\mu(E_n) = 2n < \infty$*, so* m^* *is* σ*-finite.*

Remark 2. *If whenever* $\mu(E) = \infty$*,* $E \in \mathcal{M}$*, then* $\exists F \subset E$*,* $F \in \mathcal{M}$ *such that* $\mu(F) < \infty$ *then* μ *is called a* **semi-finite measure***.*

Example 4. *Consider a non-empty* X*,* $\mathcal{M} = \mathcal{P}(X)$*, let* $f : X \to [0, \infty]$*. Define:*

$$\mu(E) = \sum_{x \in E} f(x)$$

Clearly, $\mu(\varnothing) = 0$*. Also, countable additivity holds as sums are positive and can be rearranged.*

- $f \equiv 1$ *then* $\mu(E)$ *"counts elements".*
- *Let:*

  $$f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

  But, take $A = [-1, 1]$*,* $B = [-2, 2]$*,* $X = \mathbb{R}$*,* $x_0 = 0$*. Then:*

  $$\mu(A \cup B) \neq \mu(A) + \mu(B)$$

Example 5. *Let* X *be an infinite set,* $\mathcal{M} = \mathcal{P}(X)$*. Define:*

$$\mu(E) = 0 \qquad \text{(if } E \text{ is finite)}$$
$$\mu(E) = \infty \qquad \text{(if } E \text{ is infinite)}$$

Claim. *This is not a measure but is a finitely additive measure.*

Example 6. *Let X be an infinite set, and consider:*

$$\mathcal{M} = \{E \subset X : E \text{ is countable or } E^c \text{ is countable}\}$$

Define:

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ countable} \\ \infty & \text{otherwise} \end{cases}$$

Exercise: prove this is a measure.

Theorem 1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. The following are true:*

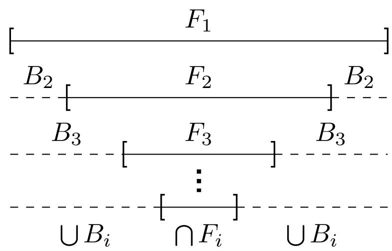
1. *Monotonicity: if $E \subset F$, $E \in \mathcal{M}, F \in \mathcal{M}$. Then $\mu(E) \leq \mu(F)$.*
2. *Subadditivity: $\{E_j\} \subset \mathcal{M}$ then:*
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$$
3. *Continuinity from below: $\{E_j\} \subset M$, $E_1 \subset E_2 \subset E_3 \subset \ldots$ Then:*
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$$
4. *Continuity from above: $\{F_j\} \subset \mathcal{M}, F_1 \supset F_2 \supset F_3 \supset \ldots$ Then:*
$$\mu\left(\bigcap_{j=1}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_j)$$

Proof.

1. $\mu(F) = \mu\left(E \uplus (F \backslash E)\right) = \mu(E) + \underbrace{\mu(F \backslash E)}_{\geq 0} \implies \mu(E) \leq \mu(F)$
2. Construct A_i such that A_i are pairwise disjoint and $\dot\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} E_i$:
$$\begin{aligned} A_1 &= E_1 \\ A_2 &= E_2 \smallsetminus E_1 = E_2 \cap E_1^c \\ &\vdots \\ A_n &= E_n \smallsetminus \bigcup_{i=1}^{n-1} E_i = E_n \cap \left(\bigcup_{i=1}^{n-1} E_i\right)^c \end{aligned}$$
The $\{A_i\}_{i=1}^{\infty}$ is clearly pairwise disjoint, $A_i \in \mathcal{M}$ $\forall i$, and:
$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\dot\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{i=1}^{\infty} \mu(A_i) \\ &\leq \sum_{i=1}^{\infty} \mu(E_i) \end{aligned}$$
3. $E_0 = \varnothing$. Define
$$\begin{aligned} A_1 &= E_1 = E_1 \smallsetminus E_0 \\ A_2 &= E_2 \smallsetminus E_1 \\ &\vdots \\ A_i &= E_i \smallsetminus E_{i-1} \end{aligned}$$

We can do this because the E_i are nested. $A_i \in \mathcal{M}$, A_i pairwise disjoint. So:

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\dot{\bigcup_{i=1}^{\infty}} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) &= \sum_{i=1}^{\infty} \mu(E_i \setminus E_{i-1}) \\
&= \sum_{i=1}^{\infty} (\mu(E_i) - \mu(E_{i-1})) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (\mu(E_i) - \mu(E_{i-1})) && \text{(telescopic sum)} \\
&= \lim_{n\to\infty} \mu(E_n) - \mu(E_0) \\
&= \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$



4. $F_1 \supset F_2 \supset ...$, define: $B_i = F_i^c \cap F_1$. So $F_1 = B_i \cup F_i$ $\forall i$, and we have:

$$F_1 = \left(\bigcap_{i=1}^{n} F_i\right) \cup \left(\bigcup_{i=1}^{n} B_i\right)$$

As we have a disjoint union for F_1, we can say that

$$\mu(F_1) = \mu\left(\bigcap_{i=1}^{\infty} F_i\right) + \mu\left(\bigcup_{i=1}^{\infty} B_i\right) \tag{*}$$

$$\mu(F_1) = \mu(F_i) + \mu(B_i) \tag{**}$$

With some playing around and by use of the diagram, we can see that $B_1 \subset B_2 \subset B_3 \subset ...$, so

$$\begin{aligned}
\mu(F_1) &= \mu\left(\bigcap_{i=1}^{\infty} F_i\right) + \lim_{i\to\infty} \mu(B_i) && \text{(by applying 3. on (*) to } \mu\left(\textstyle\bigcup_{i=1}^{\infty} B_i\right)) \\
\mu(F_1) &= \lim_{i\to\infty} \mu(F_i) + \lim_{i\to\infty} \mu(B_i) \\
\implies \lim_{i\to\infty} \mu(F_i) &= \mu\left(\bigcap_{i=1}^{\infty} F_i\right)
\end{aligned}$$

□

Definition 7. *Let $(X, \mathcal{M}, \mu)$ be a measure space. If $E \in \mathcal{M}$ satisfies $\mu(E) = 0$, then we say that E is* **null***.*

Definition 8. *If a statement about points in X is true for all $x \in X$ except for a set of measure 0, then we say that the statement is true* **almost everywhere** *(a.e.).*

Observation: $\mu(E) = 0$ and $F \subset E$, then $\mu(F) = 0$ provided $F \in \mathcal{M}$.

Definition 9. *A measure whose domain (i.e $\mathcal{M}$) contains every subset of every null set is called* **complete***.*

Example 7. *Consider $(\mathbb{R}, \mathcal{M}, \mu)$ where $\mathcal{M}$ are measurable sets and $\mu = m^*$. Then "every point in $\mathbb{R}$ is irrational" is true* **almost everywhere***.*

Theorem 2 (Solve lack of completeness of some measures). *Let $(X,\mathcal{M},\mu)$ be a measure space. Let:*

$$\mathcal{N} = \{E \in \mathcal{M} : \mu(E) = 0\}$$
$$\bar{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}, F \subset N \text{ where } N \in \mathcal{N}\}$$

Then $\bar{\mathcal{M}}$ is a σ-algebra, and there exists a **unique** *extension $\bar{\mu}$ of μ to $\bar{\mathcal{M}}$.*

Proof. We want to show firstly that $\bar{\mathcal{M}}$ is closed under

1. Countable unions
2. Complements

1. Take $A_i \in \bar{\mathcal{M}}$. We want $\bigcup_{i=1}^{\infty} A_i \in \bar{\mathcal{M}}$. We can write A_i as $A_i = E_i \cup F_i$ with $E_i \in \mathcal{M}, F_i \subset N_i, N_i \in \mathcal{N}$.

$$\bigcup_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} E_i\right) \cup \left(\bigcup_{i=1}^{\infty} F_i\right)$$

To show $\bigcup_{i=1}^{\infty} A_i \in \bar{\mathcal{M}}$, we need $\left(\bigcup_{i=1}^{\infty} E_i\right) \in \mathcal{M}$ and $\left(\bigcup_{i=1}^{\infty} F_i\right) \subset N$ with $N \in \mathcal{N}$ We know $\left(\bigcup_{i=1}^{\infty} E_i\right) \in \mathcal{M}$ because $\mathcal{M}$ is a σ-algebra. Define $N := \bigcup_{i=1}^{\infty} N_i$, so $\left(\bigcup_{i=1}^{\infty} F_i\right) \subset \bigcup_{i=1}^{\infty} N_i = N \in \mathcal{M}$, because $N_i \in \mathcal{M} \implies N \in \mathcal{M}$. We need $\mu(N) = 0$, but $\mu(N) \leq \sum_{i=1}^{\infty} \mu(N_i) = 0$ so $N \in \mathcal{N}$

2. We need $A \in \bar{\mathcal{M}} \implies A^c \in \bar{\mathcal{M}}$. Let $A = E \cup F$, $E \in \mathcal{M}, F \subset N, N \in \mathcal{N}$. w.l.o.g., $E \cap N = \varnothing$ (otherwise take $F \setminus E, N \setminus E$ instead of F and N). Need $(E \cup F)^c$ to be in $\bar{\mathcal{M}}$. We need to write $E^c \cap F^c$ as $\widetilde{E} \cup \widetilde{F}$ with $\widetilde{E} \in \mathcal{M}, \widetilde{F} \subset \widetilde{N} \in \mathcal{N}$.
Using $E \cap N = \varnothing$, we can derive the identity:

$$E \cup F = (E \cup N) \cap (N^c \cup F)$$

so:

$$(E \cup F)^c = \underbrace{(E \cup N)^c}_{\in \mathcal{M}} \cup \underbrace{(N^c \cup F)^c}_{\subset N \in \mathcal{N}}$$

We need $(E \cup N)^c \in \mathcal{M}$ and $(N^c \cup F)^c \subset Q, Q \in \mathcal{N}$.
As $E, N \in \mathcal{M} \implies (E \cup N)^c \in \mathcal{M}$. Also, $(N^c \cup F)^c = N \cap F^c \subset N$, so by taking Q = N we have that $(N^c \cup F)^c \subset Q, Q \in \mathcal{N}$

Having shown that $\bar{\mathcal{M}}$ is a σ-algebra, we want to now show there exists a unique $\bar{\mu}$. Given $A \in \bar{\mathcal{M}}$, assume $A = \underbrace{E}_{\in \mathcal{M}} \cup \underbrace{F}_{\subset N \in \mathcal{N}}$. This decomposition is not unique unfortunately. Define $\bar{\mu}(A) := \mu(E)$. It is trivial that $\bar{\mu}$ is an extension. Now w.t.s. $\bar{\mu}$ is well defined and unique. Assume $A = E_1 \cup F_1 = E_2 \cup F_2$ where $E_1, E_2 \in \mathcal{M}, F_1 \subset N_1 \in \mathcal{N}, F_2 \subset N_2 \in \mathcal{N}$. We need:

$$\mu(E_1) = \mu(E_2)$$

By sandwiching A in-between two sets that are in $\mathcal{M}$ we have

$$E_1 \subset (E_2 \cup F_2) \subset E_2 \cup N_2$$
$$\mu(E_1) \leq \mu(E_2 \cup N_2) \leq \mu(E_2) + \underbrace{\mu(N_2)}_{=0}$$

So $\mu(E_1) \leq \mu(E_2)$. By replacing $1 \leftrightarrow 2$ we have $\mu(E_2) \leq \mu(E_1)$

□

We've defined σ-algebra and measure abstractly. How do you construct σ-algebra?

Observation: an arbitrary intersection of σ-algebra (in X) is a σ-algebra.
Use of observation: Suppose E is a collection of sets in X. We can define $\mathcal{M}(E)$ as the "smallest" σ-algebra that contains E, where "smallest" is the intersection of *all* σ-algebras. ($\mathcal{M}(E)$ exists because $\mathcal{P}(X)$ is a σ-algebra containing E)

Definition 10. *(Borel σ-algebra) The smallest σ-algebra of X that contains the open sets is called the* **Borel σ-algebra**, *denoted $\mathcal{B}_X$. What we constructed in Chapter 1 is bigger (the* **Lebesgue σ-algebra**).

Proposition 8. *Let $X = \mathbb{R}$ using usual topology.*
$\mathcal{B}_{\mathbb{R}} = \mathcal{M}$*(open sets in $\mathbb{R}$) Then*

1. $\mathcal{B}_{\mathbb{R}} = \mathcal{M}$*(intervals (a,b))*
2. $\mathcal{B}_{\mathbb{R}} = \mathcal{M}([a,b])$
3. $\mathcal{B}_{\mathbb{R}} = \mathcal{M}([a,b)) = \mathcal{M}((a,b])$
4. $\mathcal{B}_{\mathbb{R}} = \mathcal{M}((a,\infty)) = \mathcal{M}((-\infty,a))$
5. $\mathcal{B}_{\mathbb{R}} = \mathcal{M}([a,\infty)) = \mathcal{M}((-\infty,a])$

Proof. Exercise. □

The motivation is that soon we will define measurable functions as "$f^{-1}(E)$ is measurable for all E which is measurable" (analogously to continuous functions on open sets in topological spaces). Having the proposition will simplify things. We will need $f^{-1}(G)$ for G in any of the smaller families.

2.1 Product spaces

How do you think of $\mathbb{R}^2$? $\mathbb{R}^2$ or $\mathbb{R}\times\mathbb{R}$? Let X_α be non empty sets, $\alpha \in \Lambda$ (in principle an uncountable index set). Consider:

$$\underline{X} := \prod_{\alpha\in\Lambda} X_\alpha$$

Define $\pi_\alpha : \underline{X} \to X_\alpha$, where π_α is the projection onto the α co-ordinate. The product σ-algebra on $\underline{X}$ is the σ- algebra generated by $\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha\}$. This is denoted by:

$$\bigotimes_{\alpha\in\Lambda} \mathcal{M}_\alpha$$

(Need to check with lecturer)

Proposition 9. *The product σ-algebra we've just defined is also the σ-algebra generated by:*

$$\prod_{\alpha\in\Lambda} E_\alpha \qquad (E_\alpha \in \mathcal{M}_\alpha)$$

provided Λ is countable.

Definition 11. *A Banach space is* **separable** *if there exists a countable subset that is dense*

Proposition 10. *(About Borel sets) Let $X_1, ..., X_n$ be metric spaces.*
$X := \prod_{i=1}^{n} X_i$ *equipped with the product metric. Then*

$$\bigotimes_{j=1}^{n} \mathcal{B}_{X_j} \subseteq \mathcal{B}_X$$

where if the X_j are all separable sets then we have equality.

2.2 Outer measures

We need to find a way to construct measures, so we have a definition.

Definition 12. *An* **outer measure** *in a (non-empty) set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies:*

1. $\mu^*(\varnothing) = 0$
2. *If $A \subset B$ then $\mu^*(A) \le \mu^*(B)$*
3. $\mu^* \left(\bigcup_{i=1}^{\infty} A_i \right) \le \sum_{i=1}^{\infty} \mu^*(A_i)$

Proposition 11. *Let $\mathcal{E} \subset \mathcal{P}(X)$, X non empty, such that $X \in \mathcal{E}, \varnothing \in \mathcal{E}$. Let $p : \mathcal{E} \to [0, \infty]$ be any function such that $p(\varnothing) = 0$. Then define for $A \in \mathcal{P}(X)$:*

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} p(E_j) : E_j \in \mathcal{E}, A \subset \bigcup_{j=1}^{\infty} E_j \right\}$$

This is an outer measure.

Proof. We need:

1. $\mu^*(\varnothing) = 0$.
   Because $p(\varnothing) = 0$, from the definition it follows that $\mu^*(\varnothing) = 0$.
2. $\mu^*(A) \le \mu^*(B)$ whenever $A \subset B$.
   This is trivial. Every cover of B is a cover of A. When you look at $\mu^*(A)$ you are taking the infimum over a bigger set, so the result follows.
3. $\mu^* \left(\bigcup_{i=1}^{\infty} A_i \right) \le \sum_{i=1}^{\infty} \mu^*(A_i)$
   Consider A_k. By properties of infimum there exists $\{E_{k,j}\}_{j=1}^{\infty}$, where $E_{k,j} \in \mathcal{E}$ such that:

$$\sum_{j=1}^{\infty} p(E_{k,j}) \le \mu^*(A_k) + \frac{\varepsilon}{2^k} \qquad (\forall \varepsilon > 0)$$

$$A_k \subset \bigcup_{j=1}^{\infty} E_{k,j}$$

   and taking unions in k gives:

$$\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} E_{k,j}$$

   (i.e. a countable cover of the set by elements in $\mathcal{E}$). This implies:

$$\mu^* \left(\bigcup_{k=1}^{\infty} A_k \right) \le \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} p(E_{k,j}) \le \sum_{k=1}^{\infty} \left(\mu^*(A_k) + \frac{\varepsilon}{2^k} \right) = \sum_{k=1}^{\infty} \mu^*(A_k) + \varepsilon$$

   By letting $\varepsilon \to 0$ we get the result.

□

Definition 13. *Let X be non-empty, μ^* an outer measure. We say that $A \subset X$ is* **measurable** *iff:*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad (\forall E \subset \mathcal{P}(X))$$

Theorem 3. *(Caratheodory) Let μ^* be an outer measure on a non-empty set X. Then the collection of measurable sets, denoted by $\mathbb{M}$ is a σ-algebra, and moreover the restriction of μ^* to $\mathbb{M}$ is a complete measure.*

Proof. We want to show that:

1. $\mathbb{M}$ is a σ-algebra i.e.:
   a) $\mathbb{M}$ is closed under complements.
   b) $\mathbb{M}$ is closed under countable unions.
2. μ is a measure, i.e.:
   a) $\mu^*(\varnothing) = 0$
   b) $\mu^*\left(\dot\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu^*(A_i)$, $A_i \in \mathbb{M}$ pairwise disjoint
3. μ is complete.

1. a) This is trivial from the definition (write A^c instead of A).
   b) We first show that $\mathbb{M}$ is an algebra, then $\mathbb{M}$ is closed under a countable disjoint union, which implies from a previous proposition that $\mathbb{M}$ is a σ-algebra.

Claim. $\mathbb{M}$ is an algebra. Consider $A, B \in \mathbb{M}$. We want $A \cup B \in \mathbb{M}$ (by induction we can then prove it for finite n (exercise)). We know that:

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad (\forall E)$$
$$\mu^*(F) = \mu^*(F \cap B) + \mu^*(F \cap B^c) \qquad (\forall F)$$

We want:

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$$

So by using the definition of A and B being measurable we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu(E \cap A^c)\\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \underbrace{\mu^*(E \cap A^c \cap B^c)}_{=\mu^*(E \cap (A \cup B)^c)}\end{aligned}$$

We want to show:

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$$

so if:

$$\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) \geq \mu^*(E \cap (A \cup B))$$

then we're done. By set theory, $A \cup B \subset ((A \cap B) \cup (A \cap B^c) \cup (A^c \cap B))$.
So $E \cap (A \cup B) \subset ((E \cap A \cap B) \cup (E \cap A \cap B^c) \cup (E \cap A^c \cap E))$, hence by properties of outer measure we've shown $A \cup B$ is measurable.
Assume $A \cap B = \varnothing$, $A, B \in \mathbb{M}$. Take $E = A \cup B$. As A is measurable:

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$
$$\Downarrow$$
$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

By induction we can show that $\mu^*\left(\dot\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \mu^*(A_i)$, with A_i pairwise disjoint. Hence we have shown that $\mathbb{M}$ is an algebra. To show it is a σ-algebra, it is enough to show $\{A_i\}_{i=1}^{\infty}$, A_i pairwise disjoint, $A_i \in \mathbb{M} \implies \dot\bigcup_{i=1}^{\infty} A_i \in \mathbb{M}$

Let $B = \dot\bigcup_{i=1}^{\infty} A_i$, $B_n = \dot\bigcup_{i=1}^{n} A_i$

We know $B_n \in \mathbb{M}$. We want $B \in \mathbb{M}$, i.e. $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$. Since $A_i \in \mathbb{M}$:

$$\mu^*(F) = \mu^*(F \cap A_n) + \mu^*(F \cap A_n^c) \qquad (\forall F)$$

Take $F = E \cap B_n$

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c)\\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1})\\ &= \mu^*(E \cap A_n) + \mu^*(E \cap A_{n-1}) + \mu^*(E \cap B_{n-2}) \text{ (inductively)}\\ &= \sum_{i=1}^{n} \mu^*(E \cap A_i)\end{aligned}$$

Details for the proof on the left

$$B_n \cap A_n = A_n$$

$$\left(\dot{\bigcup_{i=1}^{n}} A_n\right) \cap A_n = A_n$$

$$B_n \cap A_n^c = B_{n-1}$$

$$\begin{aligned}\dot{\cup}_{i=1}^{n} A_i \cap A_n^c &= \left(\dot{\bigcup_{i=1}^{n-1}} A_i \cup A_n\right) \cap A_n^c\\ &= \bigcup_{i=1}^{n-1} \underbrace{(A_i \cap A_n^c)}_{=A_i}\end{aligned}$$

$$\begin{aligned}B_n \in \mathbb{M} \Rightarrow \mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c)\\ &\geq \mu^*(E \cap B_n) + \mu^*(E \cap B^c)\\ &= \sum_{i=1}^{n} (E \cap A_i) + \mu^*(E \cap B^c)\end{aligned}$$

$$B_n \subset B$$

$$B^c \subset B_n^c$$

$$(E \cap B^c) \subset (E \cap B_n^c)$$

μ^* outer measure so

$$\mu^*(E \cap B^c) \geq \mu^*(E \cap B_n^c)$$

$$\underbrace{\text{LHS}}_{\text{indep. of } n} \geq \underbrace{\text{RHS}}_{\text{dep. of } n} \Rightarrow \text{LHS} \geq \lim_{n \to \infty} \text{RHS}$$

$$\begin{aligned}\mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c)\\ &\geq \mu^*(E \cap B) + \mu^*(E \cap B^c) \ (*) \ \text{(Reason: } E \cap B \subset \textstyle\bigcup (E \cap A_i))\end{aligned}$$

So 1 is complete.

2. a) Trivial.
   b) Rewrite * for $E = B$:

$$\mu^*\left(\dot{\bigcup_{i=1}^{\infty}} A_i\right) = \mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) + \mu^*(B \cap B^c) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) = \sum_{i=1}^{\infty} \mu^*(A_i)$$

3. We require that if $A \in \mathbb{M}, \mu^*(A) = 0$ then for every $W \subset A$ we have $W \in \mathbb{M}$. Take $W \subset A \in \mathbb{M}$. We *want*:

$$\mu^*(E) \geq \mu^*(E \cap W) + \mu^*(E \cap W^c) \qquad (\forall E)$$

$E \cap W \subset E \cap A \subset A$, so:

$$\underbrace{\mu^*(E \cap W)}_{=0} \leq \mu^*(E \cap A) \leq \mu^*(A) = 0$$

So we need $\mu^*(E) \geq \mu^*(E \cap W^c)$. This is trivial as $E \cap W^c \subset E$.

□

- Mechanism for constructing σ-algebra.
- We have a way (Caratheodory) to construct measures that come with a σ-algebra.

Need a family ξ and a family ρ
The outcome is that a measure and a σ-measure of measurable sets.
In general $\mathbb{M}(\xi)$ is not the σ-algebra of measurable sets!

- In $\mathbb{R}$, ξ = open intervals $\rightarrow \mathbb{M}(\xi)$ = Borel set.
  What comes out of Caratheodory is strictly bigger. That σ-algebra is called the Lebesgue σ-algebra, denoted by $\mathcal{L}$
  $\mathcal{L}$ is the completion of $\mathcal{B}$

Theorem 4 (Describing all possible measures in $\mathbb{R}, \mathcal{B}_{\mathbb{R}}$)**.** *If $F : \mathbb{R} \rightarrow \mathbb{R}$ is increasing, right continuous, then there exists a* **unique** *Borel measure, μ_F that satisfies $\mu_F([a,b]) = F(b) - F(a)$ (If there exists another such function G then $G = F$ + constant). Conversely, if μ is a measure in $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ that is finite on all bounded sets then the function:*

$$F(x) = \begin{cases} \mu((0,x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \mu([-x,0)) & \text{if } x < 0 \end{cases}$$

is increasing, right continuous, and $\mu_F = \mu$.

3 Measurable Functions and Integration

Let $f : X \to Y$, $(X, \mathcal{M}), (Y, \mathcal{N})$ be measurable spaces.

Definition 14. *Given $(X, \mathcal{M}), (Y, \mathcal{N})$ measure spaces, $X, Y \neq \varnothing$, we say that $f : X \to Y$ is* ***measurable*** *iff $f^{-1}(E) \in \mathcal{M}\ \forall E \in \mathcal{N}$.*

Observation: the composition of measurable functions is measurable.

$$(X, \mathcal{M}) \xrightarrow{f} (Y, \mathcal{N}) \xrightarrow{g} (Z, \mathcal{O}) \qquad (\mathcal{M}, \mathcal{N}, \mathcal{O}\ \sigma\text{-algebras in } X, Y, Z \text{ respectively})$$

Exercise: Decide whether or not this is true for $f : X \to Y$,

$$f(\bigcup E_n) = \bigcup f(E_n) \qquad f^{-1}(\bigcup E_n) = \bigcup f^{-1}(E_n)$$
$$f(\bigcap E_n) = \bigcap f(E_n) \qquad f^{-1}(\bigcap E_n) = \bigcap f^{-1}(E_n)$$
$$f(E^c) = (f(E))^c \qquad f^{-1}(E^c) = (f^{-1}(E))^c$$

Proposition 12. *Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be measurable spaces, $f : X \to Y$. Assume that $\mathcal{N}$ is generated by a collection ξ, i.e. the smallest σ-algebra $\mathcal{M}(\xi) = \mathcal{N}$ containing ξ is $\mathcal{N}$. Then:*

$$f \text{ is measurable (i.e. } f^{-1}(F) \in \mathcal{M}\ \forall F \in \mathcal{N}) \iff f^{-1}(E) \in \mathcal{M}\ \forall E \in \xi$$

Proof.

$\boxed{\Longrightarrow}$: f is measurable by definition: $f^{-1}(F) \in \mathcal{M}\ \forall F \in \mathcal{N}$.

$\boxed{\Longleftarrow}$: Look at the collection of sets, G, for which $f^{-1}(G) \in \mathcal{M}$. Let $\{G : f^{-1}(G) \in \mathcal{M}\} = \Omega$. First, $\xi \subset \Omega$. Now, Ω is a σ-algebra by *. $\mathcal{N} = \mathcal{M}(\xi) \subset \Omega$.

□

Proposition 13 (Corollary)**.** *$f : X \to \mathbb{R}$. $(X, \mathcal{M})$, $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then the following are equivalent:*

a) f *is measurable from* $(X, \mathcal{M})$ *into* $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

b) $f^{-1}((a, \infty)) \in \mathcal{M}\ \forall a$

c) $f^{-1}([a, \infty)) \in \mathcal{M}\ \forall a$

d) $f^{-1}((-\infty, c)) \in \mathcal{M}\ \forall c$

e) $f^{-1}((-\infty, c]) \in \mathcal{M}\ \forall c$

Corollary 1. *Let X, Y be metric spaces, then every continuous function is measurable from (X, β_X) to $(Y, \mathcal{B}_Y)$.*

Proof.
f continuous $\iff f^{-1}(U)$ open $\forall U$ open in Y
$f^{-1}(U)$ open $\forall U \implies f^{-1}(U)$ is in $\mathcal{B}_x \forall U$ open $\implies f$ is measurable. □

For most of this course, $Y = \mathbb{R}$ or $Y = \mathbb{C}$.

Definition 15. *Given $(X, \mathcal{M})$ measurable space with $X \neq \varnothing$ and $f : X \to \mathbb{R}$ (or $\mathbb{C}$), we say f is* $(X, \mathcal{M})$**-measurable** *if f is measurable from $(X, \mathcal{M})$ to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ (or $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$).*

Definition 16. *Let $f : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$). We say f is* **Borel measurable** *if f is measurable from $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ (or $(\mathbb{R}, \mathcal{B}_{\mathbb{C}})$).*

Definition 17. *We say f is* **Lebesgue measurable** *if f is measurable from $(\mathbb{R}^n, \mathcal{L})$ into $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ (or $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$), where $\mathcal{L}$ in $\mathbb{R}^n$ is the completion of $\mathcal{B}_{\mathbb{R}^n}$.*

Remark 3. *Composition of Borel measurable functions is Borel measurable but the composition of Lebesgue measurable functions is not necessarily Lebesgue measurable!*

Example 8. *Let* $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$. *Consider* $(f \circ g)^{-1}(E)$.

- *Clear that* E *Borel,* f *and* g *Borel measurable, then* $f^{-1}(E)$ *Borel* $\implies g^{-1}(f^{-1}(E))$ *Borel.*
- *But if* f, g *Lebesgue measurable then* E *Borel* $\implies f^{-1}(E)$ *is Lebesgue* $\implies g^{-1}(f^{-1}(E))$ *not necessarily Lebesgue.*
- *If* f *Borel,* g *Lebesgue, then* $f \circ g$ *is Lebesgue.*

Proposition 14. *Let* $(X, \mathcal{M})$, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$. *If* f, g *are measurable, then* $|f|$, $f + g$, $\min(f, g)$, $\max(f, g)$, $f \cdot g$ *are measurable. So is* cf *for* $c \in \mathbb{R}$.

Proof. By previous proposition, only need to check f^{-1} for a family that generates $\mathcal{B}_{\mathbb{R}}$.

$f + g$:

$$f^{-1}(E) \in \mathcal{M} \qquad (\forall E \in \mathcal{B}_{\mathbb{R}})$$
$$g^{-1}(E) \in \mathcal{M} \qquad (\forall E \in \mathcal{B}_{\mathbb{R}})$$
$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} (\{f > r\} \cap \{g > a - r\})$$

(Notation: $\{f > r\} = f^{-1}((r, \infty)) = \{x : f(x) > r\}$)
Therefore, $f + g$ measurable as it is a countable union of measurable sets.

cf: If $c = 0$:

$$(cf)^{-1}((a, \infty)) = \begin{cases} X & \text{if } a < 0 \\ \varnothing & \text{otherwise} \end{cases}$$

If $c > 0$:

$$(cf)^{-1}((a, \infty)) = \{cf > a\} = \left\{f > \frac{a}{c}\right\}$$

$|f|$: $(|f|)^{-1}((a, \infty)) = \{f > a\} \cup \{f < -a\}$

$\min(f, g)$: $\{\min(f, g) > a\} = \{f > a\} \cap \{g > a\}$

$\max(f, g)$: $\{\max(f, g) > a\} = \{f > a\} \cup \{g > a\}$

$f \cdot g$: Show f^2 is measurable (exercise).

$$(f + g)^2 = f^2 + g^2 + 2f \cdot g \implies f \cdot g = \frac{(f + g)^2 - f^2 - g^2}{2}$$

□

Proposition 15. *Let* $f_j : X \to \mathbb{R} \cup \{-\infty, \infty\}$, f_j *measurable. Then* $\sup_j f_j$ *and* $\inf_j f_j$ *are measurable. So are* $\liminf_{j \to \infty} f_j$, $\limsup_{j \to \infty} f_j$, *and* $\lim_{j \to \infty} f_j$.

Proof.

$$\left\{\inf_j f_j < a\right\} = \bigcup_{j=1}^{\infty} \{f_j < a\}$$
$$\left\{\sup_j f_j > a\right\} = \bigcup_{j=1}^{\infty} \{f_j > a\}$$
$$\liminf_{j \to \infty} f_j = \sup_k \inf_{j \geq k} f_j$$
$$\limsup_{j \to \infty} f_j = \inf_k \sup_{j \geq k} f_j$$
$$\lim_{j \to \infty} f_j = \liminf_{j \to \infty} f_j = \limsup_{j \to \infty} f_j \qquad \text{(where it exists)}$$

□

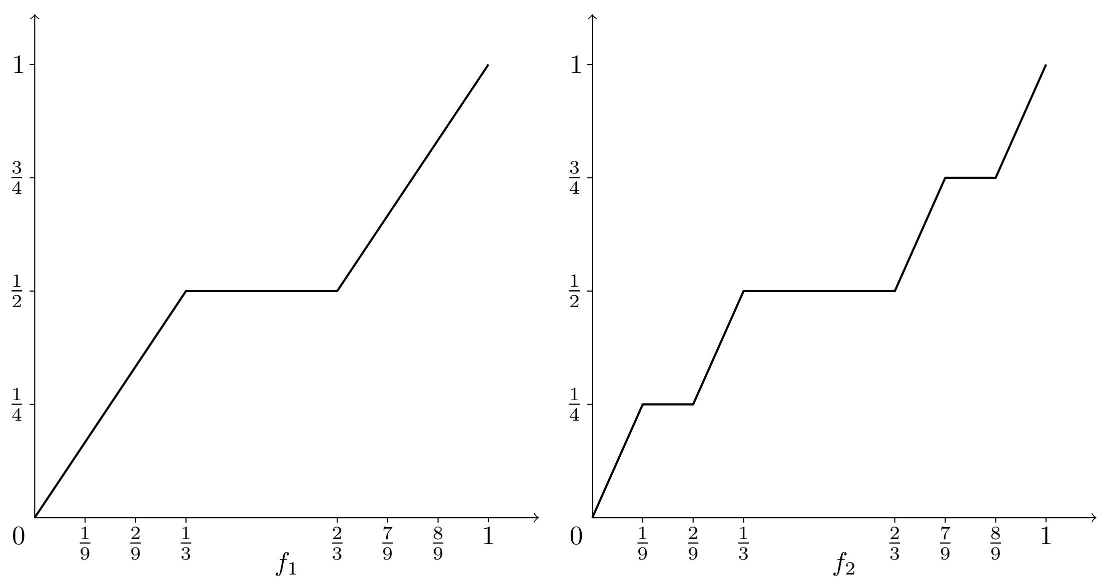
Alternate proof. Let $f := \lim_{j\to\infty} f_j$

$$\{f > a\} = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N} \left\{f_n > a + \frac{1}{m}\right\}$$

$\square$

Remark 4. *We need not define a measure to talk about measurable functions (like continuous functions in topology).*

3.1 Devil's staircase



This procedure provides $\{f_n\}_{n=1}^{\infty}$. Let $f := \lim_{n\to\infty} f_n$. f_n are continuous and uniformly continuous (exercise). Properties:

- $f(0) = 0$, $f(1) = 1$.
- f' exists on every open interval we remove (and equals 0).
- $f(1) - f(0) \neq \int_0^1 f'(x)\,dx$.
- The derivative is 0 on a set of measure 1.
- f(Cantor set C) has measure 1. f(complement of C) $\subset \{\frac{c}{2^n} : c \in \{0, 1, ..., 2^n\}\} \subset \mathbb{Q}$, so has measure 0.
- Except for the points $\{\frac{c}{2^n}\}$ the function has an inverse. Let $x \in [0,1]$ be written in base 3 as:

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n} \qquad (\varepsilon \in \{0,1,2\})$$

Cantor set points are where $\varepsilon_n \in \{0,2\}$ $\forall n$. Then:

$$f(x) = \begin{cases} \frac{1}{2}\sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n} & \text{if } x \in C \\ \text{unique constant such that } f \text{ is continuous} & \text{otherwise} \end{cases}$$

As for the inverse of f (call it g), write:

$$y = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n} \qquad (\varepsilon_n = 0, 1)$$

Now:

$$g(y) = \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n} \qquad \text{(for } y \neq \{\tfrac{c}{2^n}\}\text{)}$$

g maps $\underbrace{[0,1] \setminus \{\frac{c}{2^n}\}}_{\text{Inside is a measurable set, say } E.}$ into the Cantor set. $g(E) \subset C$ so it is measurable. $g^{-1}(g(E)) = E$, so the inverse of a measurable set $g(E)$ is not measurable (g, which arose as the inverse of a continuous function, is *not* Borel measurable).

3.2 Integration for $f : X \to [0, \infty]$

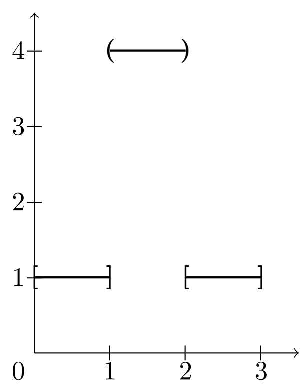
Definition 18. *Let $(X, \mathcal{M})$ be a measurable space. χ_E is a* **characteristic function** *(***indicator function***) if:*

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in E^c \end{cases}$$

Definition 19. *A function f is* **simple** *if it takes finitely many values. i.e. if $\exists N \in \mathbb{N}$ and $E_1, ..., E_N \subset X$ such that $f(x) = \sum_{j=1}^N \alpha_j \chi_{E_j}(x)$.*

Definition 20. *$f : X \to \mathbb{R}$ (or $\mathbb{C}$) is* **simple measurable** *if $\exists N \in \mathbb{N}$, $\{E_1, ..., E_N\} \subset \mathcal{M}$ such that $f = \sum_{j=1}^N \alpha_j \chi_{E_j}$.*

Definition 21. *For a simple function f we say it is written in* **standard form** *if $E_j = f^{-1}(\alpha_j)$.*



$$\begin{aligned} f(x) &= 1\chi_{[0,1]} + 4\chi_{(1,2)} + 1\chi_{[2,3]} \\ &= 1\chi_{[0,3]} + 3\chi_{(1,2)} \\ &= \underbrace{1\chi_{[0,1]\cup[2,3]} + 4\chi_{(1,2)}}_{\text{standard form}} \end{aligned}$$

Theorem 5. *Let $(X, \mathcal{M})$ be a measurable space, $f : (X, \mathcal{M}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ for $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$*

a) If $f : X \to [0, \infty]$ measurable, then there is a sequence $\{\phi_i\}_{i=1}^\infty$ of simple measurable functions such that:

$$0 \leq \phi_1 \leq \phi_2 \leq ... \leq f$$

and such that $\lim_{i\to\infty} \phi_i(x) = f(x)$ $\forall x \in X$. Moreover, $\phi_i \to f$ uniformly on any subset where f is bounded.

b) If $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is measurable, then there is a sequence $\{\phi_j\}_{j=1}^\infty$ of measurable simple functions such that:

$$0 \leq |\phi_1| \leq |\phi_2| \leq ... \leq |f|$$

and such that $\lim_{j\to\infty} \phi_j(x) = f(x)$ $\forall x \in X$. Moreover, $\phi_j \to f$ uniformly on any subset where f is bounded.

Notation:

- $\phi_j \rightrightarrows f$ means ϕ_j converges to f uniformly.
- a.e. means **almost everywhere**.

Proof.

a) For $n = 1, 2, ...$, define:

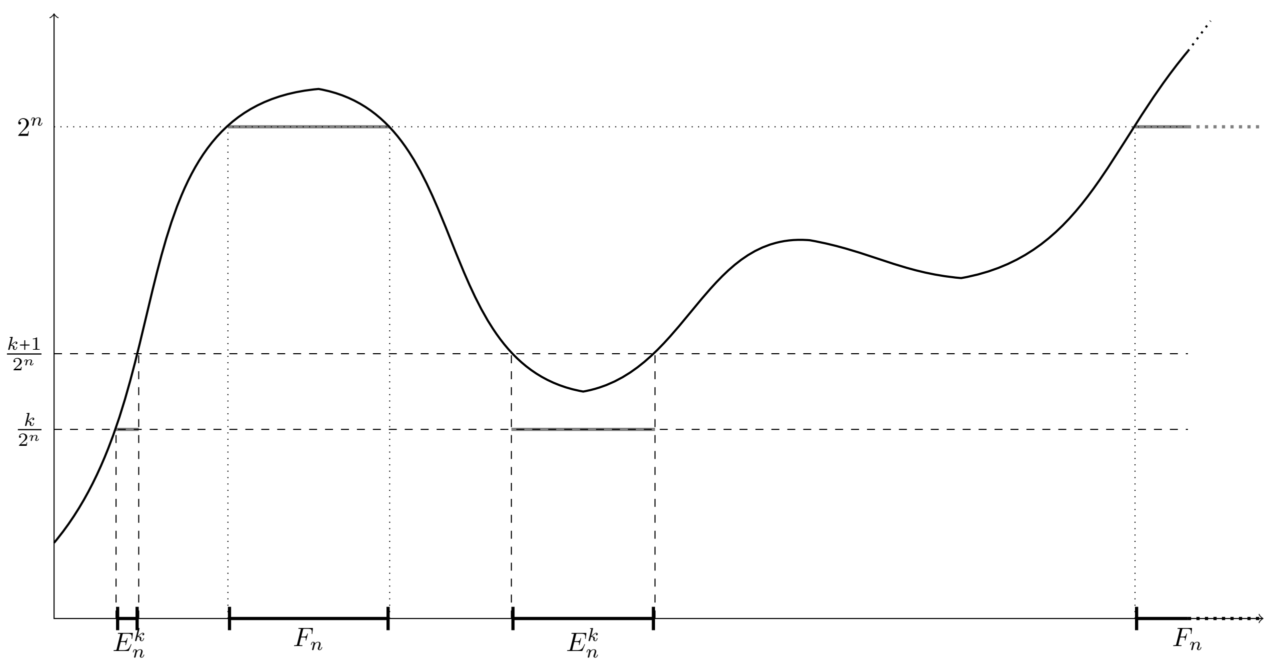
$$E_n^k = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) \qquad (k = 0, 1, ..., 2^{2n-1})$$
$$F_n = f^{-1}([2^n, \infty))$$

Then $E_n^k, F_n \subset X$. Define:

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n}\chi_{E_n^k} + 2^n \chi_{F_n}$$

Claim.

- $\phi_j \leq \phi_{j+1} \leq f$, $\forall j = 1, 2, ...$
- $\phi_j \to f$, $\phi \rightrightarrows f$ on f bounded.



b) $f : X \to \mathbb{R}$ (or $\mathbb{C}$). Define:

$$f^+ = \max\{f, 0\}$$
$$f^- = \max\{-f, 0\}$$

Now, $f^+ + f^- = |f|$ and $f^+ - f^- = f$, f^+ and f^- are measurable, $f^+, f^- : X \to [0, \infty]$. To each one, apply part a) to get $\{\phi_n^+\}, \{\phi_n^-\}$. Define $\phi_n = \phi_n^+ - \phi_n^-$. For $f : X \to \mathbb{C}$, $f = \Re(f) + i\Im(f)$ with $\Re(f), \Im(f) : X \to \mathbb{R}$ and apply the real case to each.

□

Proposition 16. *Let $(X, \mathcal{M}, \nu)$, ν complete, $f : X \to \mathbb{R}$ (or $\mathbb{C}$) be a measurable function ($\mathcal{B}_\mathbb{R}$ or $\mathcal{B}_\mathbb{C}$). If $g = f$ a.e. then g is measurable.*

Proposition 17. *Let $(X, \mathcal{M}, \nu)$, ν complete, $f_n \to f$ a.e. with f_n measurable. Then f is also measurable.*

Define:

$$\mathcal{L}^+ := \{f : f : X \to [0,\infty],\ f \text{ is measurable}\}$$

Let $(X, \mathcal{M}, \nu)$ and $f \in \mathcal{L}^+$, then:

$$\int f\, d\nu := \sum_{k=1}^{N} a_k \nu(A_k)$$

For $f = \sum_{k=1}^{N} a_k \chi_{A_k}$ in standard representation. Also, for A measurable:

$$\int_A f\, d\nu := \int \underbrace{f \cdot \chi_A}_{\text{simple}}\, d\nu$$

Proposition 18. *The integral of a simple function $f = \sum_1^N a_n \mathbb{1}_{A_n}$ where A_n are disjoint is $\int f d\mu := \sum a_n \mu A_n$.*

Proof. See Notes from Jose. ☐

Proposition 19. *If $f = \sum_1^N b_j \mathbb{1}_{B_j}$, then $\int f d\mu = \sum b_j \mathbb{1}_{B_j}$*

Proof. Let the measurable sets $\{C_{i i\in[m]}\}$ be the unique coarsest partition of $\bigcup B_j$ such that for any $j \in [N]$, we can write each nonempty B_j as a disjoint union of C_i.
For each i, let $I(i)$ be the unique indexing set such that $j \in I(i)$ iff $C_i \subset B_j$. Thus by construction,

$$\bigsqcup_{i:j\in I(i)} C_i = B_j$$

And also

$$f = \sum_{j=1}^{N} b_j \mathbb{1}_{B_j} = \sum_{i=1}^{M} \left(\sum_{j\in I(i)} b_j \right) \mathbb{1}_{C_i}$$

Then

$$\int f d\mu = \sum_{i=1}^{M} \left(\sum_{j\in I(i)} b_j \right) \mu(C_i) = \sum_{j=1}^{N} b_j \sum_{i:j\in I(i)} \mu(C_i) = \sum_{j=1}^{N} b_j \mu(B_j)$$

☐

Proposition 20. *Let $f \in \mathcal{L}^+$, $f = \sum_{j=1}^{M} b_j \chi_{B_j}$, not necessarily in standard form, then:*

$$\int f\, d\nu = \sum_{j=1}^{M} b_j \nu(B_j)$$

Proof. Let $f = \sum_{k=1}^{N} a_k \chi_{A_k}$ in standard form.

$$\int f\, d\nu = \sum_{k=1}^{N} a_k \nu(A_k)$$

Assume one $a_k = 0$ and one of $b_j = 0$, so that:

$$\dot{\bigcup_{k=1}^{N}} A_k = X = \bigcup_{j=1}^{M} B_j$$

$\{B_j\}$ may not be disjoint, but we know $\{A_k\}$ are.

$$B_j = B_j \cap X = B_j \cap \left(\dot{\bigcup_{k=1}^{N}} A_k \right) = \dot{\bigcup_{k=1}^{N}} (B_j \cap A_k)$$

Therefore:

$$\begin{aligned}\sum_{j=1}^{M} b_j \nu(B_j) &= \sum_{j=1}^{M} b_j \nu\left(\dot{\bigcup_{k=1}^{N}} B_j \cap A_k\right)\\ &= \sum_{j=1}^{M}\sum_{k=1}^{N} b_j \nu(B_j \cap A_k)\\ &= \sum_{j=1}^{M}\sum_{k=1}^{N} a_k \nu(B_j \cap A_k)\\ &= \sum_{k=1}^{N} a_k \sum_{j=1}^{M} \nu(B_j \cap A_k)\end{aligned}$$

Assume for now that $\{B_j\}$ are pairwise disjoint, then:

$$\begin{aligned}\int f\,d\mu = \sum_{k=1}^{N} a_k \mu(A_k) &= \sum_{k=1}^{N}\sum_{j=1}^{M} a_k \mu(A_k \cap B_j)\\ &= \sum_{j=1}^{M}\sum_{k=1}^{N} b_j \mu(A_k \cap B_j)\\ &= \sum_{j=1}^{N} b_j \mu(B_j)\end{aligned}$$

The proof pauses here. □

Proposition 21. *Let ϕ, ψ be measurable, simple, nonnegative, then:*

1. $\int c\phi\,d\mu = c\int \phi\,d\mu$ *$(c > 0)$*
2. $\int \phi + \psi\,d\mu = \int \phi\,d\mu + \int \psi\,d\mu$
3. *If $\phi(x) \le \psi(x)$ $\forall x \in X$ then $\int \phi\,d\mu \le \int \psi d\mu$.*
4. *Fix ϕ, then the map $A \mapsto \int_A \phi\,d\mu$ is a measure $\forall A \in \mathcal{M}$. Call it ν.*

Proof.

1. Trivial as $c\phi$ is simple. So $\int \phi\,d\mu = \sum_{k=1}^{N} a_k \mu(A_k)$ in standard form. Then:

$$\int c\phi\,d\mu = \sum_{k=1}^{N} c a_k \mu(A_k) = c\sum_{k=1}^{N} a_k \mu(A_k) = c\int \phi\,d\mu$$

2. Assume:

$$\left.\begin{aligned}\phi &= \textstyle\sum_{k=1}^{N} a_k \chi_{A_k}\\ \psi &= \textstyle\sum_{j=1}^{M} b_j \chi_{B_j}\end{aligned}\right\}\text{in standard form}$$

and assume $\biguplus_{k=1}^{N} A_k = X = \biguplus_{j=1}^{M} B_j$:

$$\begin{aligned}
\int \phi \, d\mu + \int \psi \, d\mu &= \sum_{k=1}^{N} a_k \mu(A_k) + \sum_{j=1}^{M} b_j \mu(B_j) \\
&= \sum_{k=1}^{N} a_k \mu\left(A_k \cap \biguplus_{j=1}^{M} B_j\right) + \sum_{j=1}^{M} b_j \left(B_j \cap \biguplus_{k=1}^{N} A_k\right) \\
&= \sum_{k=1}^{N} a_k \mu\left(\biguplus_{j=1}^{M} (A_k \cap B_j)\right) + \sum_{j=1}^{M} b_j \mu\left(\biguplus_{k=1}^{N} (B_j \cap A_k)\right) \\
&= \sum_{k=1}^{N} \sum_{j=1}^{M} a_k \mu(A_k \cap B_j) + \sum_{j=1}^{M} \sum_{k=1}^{N} b_j \mu(B_j \cap A_k) \\
&= \sum_{j=1}^{M} \sum_{k=1}^{N} (a_k + b_j) \mu(A_k \cap B_j)
\end{aligned}$$

Now:

$$\begin{aligned}
\phi &= \sum_{k=1}^{N} a_k \chi_{A_k} = \sum_{k=1}^{N} a_k \chi_{A_k \cap \biguplus_{j=1}^{M} B_j} = \sum_{k=1}^{N} a_k \chi_{\biguplus_{j=1}^{M} (A_k \cap B_j)} = \sum_{k=1}^{N} a_k \sum_{j=1}^{M} \chi_{A_k \cap B_j} \\
\psi &= \sum_{j=1}^{M} b_j \chi_{B_j} = \sum_{j=1}^{M} b_j \chi_{B_j \cap \biguplus_{k=1}^{N} A_k} = \sum_{j=1}^{M} b_j \chi_{\biguplus_{k=1}^{N} (A_k \cap B_j)} = \dots \\
\implies \phi + \psi &= \sum_{j=1}^{M} \sum_{k=1}^{N} (a_k + b_j) \chi_{A_k \cap B_j}
\end{aligned}$$

So $\int \phi \, d\mu + \int \psi \, d\mu = \int \phi + \psi \, d\mu$.

□

Back to our original proof for a bit:

Proof. Let $\phi = \sum_{i=1}^{T} c_i \chi_{E_i}$ be measurable simple. Let:

$$\phi = \phi_1 + \phi_2 + \dots + \phi_T$$

where $\phi_i = c_i \chi_{E_i}$ is in standard form. Then:

$$\begin{aligned}
\int \phi \, d\mu = \int \phi_1 + \phi_2 + \dots + \phi_T \, d\mu &= \int \phi_1 \, d\mu + \dots + \int \phi_T \, d\mu \\
&= \sum_{i=1}^{T} \int \phi_i \, d\mu \\
&= \sum_{i=1}^{T} c_i \int \chi_{E_i} \, d\mu \\
&= \sum_{i=1}^{T} c_i \mu(E_i)
\end{aligned}$$

□

Back to the proof of the proposition:

Proof.

3. $\phi(x) = \sum_{k=1}^{N} a_k \chi_{A_k}$ and $\psi(x) = \sum_{j=1}^{M} b_j \chi_{B_j}$ in standard form, $\dot\bigcup_{k=1}^{N} A_k = X = \dot\bigcup_{j=1}^{M} B_j$. From previous argument:

$$\phi(x) = \sum_{k=1}^{N} \sum_{j=1}^{M} a_k \chi_{A_k \cap B_j}$$
$$\psi(x) = \sum_{k=1}^{N} \sum_{j=1}^{M} b_j \chi_{A_k \cap B_j}$$

$\{A_k \cap B_j\}$ are pairwise disjoint. Consider $x \in A_k \cap B_j \neq \varnothing$ for some i, j.

$$\phi(x) = a_k, \psi(x) = b_j \implies a_k \leq b_j$$

So:

$$\int \phi \, d\mu = \sum_{j=1}^{M} \sum_{k=1}^{N} a_k \mu(A_k \cap B_j) \leq \sum_{k=1}^{M} \sum_{j=1}^{M} b_j \mu(A_k \cap B_j) = \int \psi \, d\mu$$

4. Need:
   - $\nu(\varnothing) = 0$
   - $\nu(\dot\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \nu(A_i)$

   Where $A = \dot\bigcup_{i=1}^{\infty} A_i$. Write $\phi = \sum_{j=1}^{M} b_j \chi_{B_j}$ standard form with $\dot\bigcup_{j=1}^{M} B_j = X$.

$$\begin{aligned}
\int_A \phi \, d\mu = \int \phi \chi_A \, d\mu &= \int \sum_{j=1}^{M} b_j \chi_{B_j} \chi_A \, d\mu \\
&= \int \sum_{j=1}^{M} b_j \chi_{B_j \cap A} \, d\mu \\
&= \sum_{j=1}^{M} b_j \mu(B_j \cap A) \\
&= \sum_{j=1}^{M} b_j \mu\left(B_j \cap \dot\bigcup_{i=1}^{\infty} A_i\right) = \sum_{j=1}^{M} b_j \mu\left(\dot\bigcup_{i=1}^{\infty} (B_j \cap A_i)\right) \\
&= \sum_{j=1}^{M} b_j \mu\left(\lim_{n\to\infty} \dot\bigcup_{i=1}^{n} (B_j \cap A_i)\right) \\
&= \sum_{j=1}^{M} b_j \lim_{n\to\infty} \mu\left(\dot\bigcup_{i=1}^{n} (B_j \cap A_i)\right) \\
&= \sum_{j=1}^{M} b_j \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_j \cap A_i) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{M} b_j \mu(B_j \cap A_i) \\
&= \sum_{i=1}^{\infty} \int \sum_{j=1}^{M} b_j \chi_{B_j \cap A_i} \, d\mu \\
&= \sum_{i=1}^{\infty} \int \sum_{j=1}^{M} b_j \chi_{B_j} \chi_{A_i} \, d\mu \\
&= \sum_{i=1}^{\infty} \int_{A_i} \phi \, d\mu = \sum_{i=1}^{\infty} \nu(A_i)
\end{aligned}$$

□

Define $\mathcal{L}^+ = \{$f: f is measurable and non-negative$\}$. Then we define $\int f\,d\mu := \sup\{\int \phi\,d\mu : \phi$ simple measurable, $0 \le \phi \le f\}$

$$\int f\,d\mu \le \int g\,d\mu \qquad \text{(if } f \le g,\ f, g \in \mathcal{L}^+\text{)}$$
$$\int cf\,d\mu = c\int f\,d\mu \qquad \text{(for } c \ge 0\text{)}$$

But we don't know that $\int f + g\,d\mu = \int f\,d\mu + \int g\,d\mu$

Theorem 6 (Monotone convergence)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n\}_{n=1}^\infty \subset \mathcal{L}^+$. If $f_n \le f_{n+1}\ \forall n$, then:*

$$\lim_{n\to\infty}\int f_n\,d\mu = \int \lim_{n\to\infty} f_n\,d\mu$$

Proof. First, $\lim_{n\to\infty} f_n$ exists as (f_n) is a monotone sequence in $\mathbb{R}^+$ and measurable by an earlier theorem.

$$f_m \le \lim_{n\to\infty} f_n \implies \int f_m\,d\mu \le \int \lim_{n\to\infty} f_n\,d\mu \qquad (\forall m \in \mathbb{N})$$

Take limits to get:

$$\lim_{m\to\infty}\int f_m\,d\mu \le \int \lim_{n\to\infty} f_n\,d\mu$$

Fix ϕ as a simple measurable function:

$$0 \le \phi \le f = \lim_{n\to\infty} f_n$$

Fix $\alpha \in (0,1)$. Define $E_n = \{x \in X : f_n(x) \ge \alpha\phi(x)\}$. Claim that $\bigcup_{n=1}^\infty E_n = X$ (because $f_n(x) \nearrow f(x)$ and $\alpha\phi(x) < f(x)$ for $f(x) \ne 0$). Consider:

$$\lim_{n\to\infty}\int_{E_n}\phi\,d\mu = \int_X \phi\,d\mu$$
$$\lim_{n\to\infty}\nu(E_n) \underbrace{=}_{\text{by continuity}} \nu\left(\bigcup_{n=1}^\infty E_n\right) = \nu(X)$$

(With vertical equals signs to add)
We know:

$$\int_{E_n}\alpha\phi\,d\mu \le \int_{E_n} f_n\,d\mu \le \int_X f_n\,d\mu \le \int_X f_{n+1}\,d\mu \le \lim_{n\to\infty}\int_X f_n\,d\mu$$

Thus:

$$\lim_{m\to\infty}\int_{E_m}\alpha\phi\,d\mu \le \lim_{n\to\infty}\int f_n\,d\mu$$

So:

$$\begin{aligned}
&\lim_{n\to\infty}\int f_n\,d\mu \ge \int \alpha\phi\,d\mu\\
\implies &\lim_{n\to\infty}\int f_n\,d\mu \ge \sup_\phi\left\{\int \alpha\phi\,d\mu : 0 \le \phi \le f\right\}\\
&= \int \alpha f\,d\mu\\
&= \alpha\int f\,d\mu\\
\implies &\lim_{n\to\infty}\int f_n\,d\mu \ge \sup_\alpha\left\{\alpha\int f\,d\mu\right\}\\
&= \int f\,d\mu
\end{aligned}$$

□

Example 9 (Counterexamples)**.** *Let:*

- $f_n = \chi_{[n,n+1]}$, $f_n(x) \to 0 \ \forall x$.
- $g_n = n\chi_{(0,\frac{1}{n})}$, $g_n(x) \to 0 \ \forall x$.

$\int f_n = 1$ and $\int g_n = 1 \ \forall n$, but $\int f = 0$ and $\int g = 0$.

The monotone convergence theorem (MCT) implies we do not need to take the supremum over all $0 \le \phi \le f$ in the definition of $\int f \, d\mu$. It's enough to take *one* family $\{\phi_n\}_{n=1}^{\infty}$ of simple measurable functions such that $0 \le \phi_n \le \phi_{n+1}$ and $\phi_n(x) \to f(x) \ \forall x$.

$$\text{MCT} \implies \lim_{n\to\infty} \int \phi_n \, d\mu = \int \lim_{n\to\infty} \phi_n \, d\mu = \int f \, d\mu$$

Recall that we proved a theorem that shows the existence of at least one such family $\{\phi_n\}_{n=1}^{\infty}$.

Theorem 7. *$(X, \mathcal{M}, \mu)$, $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^+$ then:*

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

(this proves that $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$)

Proof. Take:

- $\{\phi_n\}_{n=1}^{\infty}$ measurable simple, $0 \le \phi_n \le \phi_{n+1} \ \forall n$, $\phi_n \nearrow f$.
- $\{\psi_n\}_{n=1}^{\infty}$ measurable simple, $0 \le \psi_n \le \psi_{n+1} \ \forall n$, $\psi_n \nearrow g$.

We know, by the MCT:

$$\lim_{n\to\infty} \int \phi_n \, d\mu = \int f \, d\mu$$
$$\lim_{n\to\infty} \int \psi_n \, d\mu = \int g \, d\mu$$

Also, $\phi_n + \psi_n \nearrow f + g$, so:

$$\lim_{n\to\infty} \int (\phi_n + \psi_n) \, d\mu = \int (f + g) \, d\mu$$

We also know that:

$$\int \phi_n + \psi_n \, d\mu = \int \phi_n \, d\mu + \int \psi_n \, d\mu$$

So now just need to check:

$$\lim_{n\to\infty} \left(\int \phi_n \, d\mu + \int \psi_n \, d\mu \right) = \lim_{n\to\infty} \int \phi_n \, d\mu + \lim_{n\to\infty} \int \psi_n \, d\mu$$

As everything is non-negative, it is true by Analysis I. So, by induction:

$$\int \sum_{n=1}^{N} f_n \, d\mu = \sum_{n=1}^{N} \int f_n \, d\mu$$

Let $g_N = \sum_{n=1}^{N} f_n \, d\mu$. We have:

$$f_n \ge 0 \implies g_N \le g_{N+1}$$

and:

$$g_N \nearrow \sum_{n=1}^{\infty} f_n$$

By MCT:

$$\begin{aligned}
\int \lim_{N\to\infty} g_N \, d\mu &= \lim_{N\to\infty} \int g_N \, d\mu \\
\implies \int \sum_{n=1}^{\infty} f_n \, d\mu &= \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n \, d\mu \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n \, d\mu \\
&= \sum_{n=1}^{\infty} \int f_n \, d\mu
\end{aligned}$$

□

Theorem 8. *$(X, \mathcal{M}, \mu)$, $f \in \mathcal{L}^+$. Then:*

$$\int f \, d\mu = 0 \iff f = 0 \ a.e.$$

Proof. Notice that the statement is trivial, for measurable simple functions $\phi = \sum_{n=1}^{N} a_n \chi_{E_n}$, $a_k \geq 0$, as we know:

$$\int \phi \, d\mu = \sum_{n=1}^{N} a_n \mu(E_n)$$

and:

$$\sum_{k=1}^{N} a_n \mu(E_n) = 0 \iff \phi = 0 \text{ a.e.}$$

as:

$$\begin{aligned}
\sum_{k=1}^{N} a_n \mu(E_n) = 0 &\iff a_n \mu(E_n) = 0 && (\forall n \in \{1, ..., N\}) \\
&\iff \text{either } a_n = 0 \text{ or } \mu(E_n) = 0 && (\forall n \in \{1, ..., N\})
\end{aligned}$$

So $\{x : \phi(x) \neq 0\}$ is a finite union of sets of measure 0. Now we look at general $f \in \mathcal{L}^+$.

$$\int f \, d\mu = \sup\left\{ \int \phi \, d\mu : 0 \leq \phi \leq f, \ \phi \text{ simple measurable} \right\}$$

Suppose:

$$\begin{aligned}
f = 0 \text{ a.e.} &\implies \forall \phi \leq f \text{ measurable simple, we have } \phi = 0 \text{ a.e.} \\
&\implies \text{for those } \phi, \ \textstyle\int \phi \, d\mu = 0 && \text{(exercise)} \\
&\implies \int f \, d\mu = \sup\{0\} = 0
\end{aligned}$$

Next, want to show $\int f \, d\mu = 0 \implies f = 0$ a.e.
Look at:

$$\{x : f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \underbrace{\left\{ f \geq \frac{1}{n} \right\}}_{E_n}$$

Then:

$$\underbrace{\int f \, d\mu}_{=0} \geq \int_{E_n} f \, d\mu \geq \int_{E_n} \frac{1}{n} \, d\mu = \int \frac{1}{n} \chi_{E_n} \, d\mu = \frac{1}{n} \mu(E_n)$$

$\implies \mu(E_n) = 0$
$\implies \mu(\{x : f(x) \neq 0\}) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$ □

Corollary 2. *$\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^+$, $f_n \leq f_{n+1}$ a.e. Then:*

$$\int \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

(By $\lim_{n\to\infty} f_n$ we mean the $\lim_{n\to\infty} f_n(x)$ where it exists and 0 otherwise)

Proof. $f_n \nearrow f$ in E with $\mu(E^c) = 0$.

$$\int_E f_n \, d\mu = \int f_n \, d\mu$$
$$\int_E f \, d\mu = \int f \, d\mu$$
$$\underbrace{\int f \, d\mu}_{=\int \lim_{n\to\infty} f_n \, d\mu} = \int_E f \, d\mu = \int_E \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_E f_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

□

Warning: in general, we do *not* have:

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu$$

but...

Lemma 2 (Fatou's Lemma)**.** *Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^+$, $(X, \mathcal{M}, \mu)$, then:*

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Proof. Recall $\liminf_{n\to\infty} = \sup_k \inf_{n\geq k} f_n$. $\inf_{n\geq k} f_n \leq f_j$ for every $j \geq k$, which implies:

$$\int \inf_{n\geq k} f_n \, d\mu \leq \int f_j \, d\mu \qquad (\forall j \geq k)$$

(Let $g_k = \inf_{n\geq k} f_n$, then g_k monotone increasing)

$$\int \inf_{n\geq k} f_n \, d\mu \leq \inf_{j\geq k} \int f_j \, d\mu$$
$$\lim_{k\to\infty} \int g_k \, d\mu \leq \lim_{k\to\infty} \inf_{j\geq k} \int f_j \, d\mu$$
$$\underset{\text{MCT}}{\Longrightarrow} \int \lim_{k\to\infty} \inf_{n\geq k} f_n \, d\mu \leq \lim_{k\to\infty} \inf_{j\geq k} \int f_j \, d\mu$$
$$\Longrightarrow \int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

□

Corollary 3. *$(X, \mathcal{M}, \mu)$, $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^+$, assume $\lim_{n\to\infty} f_n = f$ a.e. Then:*

$$\int f \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Proof. Exercise. □

3.3 Integration for general $f : X \to \mathbb{R} \cup \{\pm\infty\}$

Definition 22. *Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ measurable, $f = f^+ - f^-$ with $f^+, f^- \geq 0$. Then assuming at most one of $\int f^+ \, d\mu$, $\int f^- \, d\mu$ is infinite, we define:*

$$\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu$$

Definition 23. *We say f is* **integrable** *iff $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are finite. This is equivalent to $\int |f| \, d\mu < \infty$.*

Proposition 22. *The space of integrable functions is a vector space. The integral is a linear functional in that vector space.*

Proof. If f, g are integrable then $af + bg$ integrable $\forall a, b \in \mathbb{R}$. As $|af + bg| \leq |a||f| + |b||g|$, we have:

$$\begin{aligned} \int |af + bg| \, d\mu &\leq \int |a||f| \, d\mu + \int |b||g| \, d\mu \\ &= |a| \int |f| \, d\mu + |b| \int |g| \, d\mu < \infty \end{aligned}$$

A **functional** is a map from a space of functions to $\mathbb{R}$ (or $\mathbb{C}$). So define:

$$I(f) := \int f \, d\mu$$

We need:

1. $I(af) = aI(f)$
2. $I(f + g) = I(f) + I(g)$

1.

$$\begin{aligned} I(af) &= \int af \, d\mu && (a \in \mathbb{R}) \\ &= \int (af)^+ \, d\mu - \int (af)^- \, d\mu \end{aligned}$$

There are three cases to this:

Case 1. $a = 0$, then trivial as both sides are null.

Case 2. $a > 0$, then $(af)^+ = a(f)^+$ and $(af)^- = a(f)^-$, which implies:

$$\begin{aligned} I(af) &= \int af^+ \, d\mu - \int af^- \, d\mu \\ &= a \int f^+ \, d\mu - a \int f^- \, d\mu \\ &= a \int f \, d\mu \end{aligned}$$

Case 3. $a < 0$, exercise.

2. Let $h = f + g$. Then:

$$\begin{aligned} f &= f^+ - f^- \\ g &= g^+ - g^- \\ h &= h^+ - h^- = f^+ - f^- + g^+ - g^- \end{aligned}$$

$$\begin{aligned} \implies \quad & h^+ + f^- + g^- = f + g^+ + h^- \\ \implies \quad & \int h^+ + f^- + g^- \, d\mu = \int f^+ + g^+ + h^- \, d\mu \\ \implies \quad & \int h^+ \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu = \int f^+ \, d\mu + \int g^+ \, d\mu + \int h^- \, d\mu \quad \text{(as everything is positive)} \\ \implies \quad & \int h^+ \, d\mu - \int h^- \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu + \int g^+ \, d\mu - \int g^- \, d\mu \end{aligned}$$

□

Remark 5. *For $f : X \to \mathbb{C}$, define:*

$$\int f \, d\mu := \int \Re(f) \, d\mu + i \int \Im(f) \, d\mu$$

Then, as long as everything is finite, everything translates to the complex case.

Notation: $(X, \mathcal{M}, \mu)$, we define $\mathcal{L}^1(X, \mathcal{M}, \mu)$ (a.k.a. $\mathcal{L}^1(\mu)$ or $\mathcal{L}^1(X)$) to be the space of integrable functions ($\int |f| \, d\mu < \infty$).

Proposition 23. *$(X, \mathcal{M}, \mu)$, $f : X \to \mathbb{R}$ (or $\mathbb{C}$):*

$$\left| \int f \, d\mu \right| \le \int |f| \, d\mu$$

Proof.

($\mathbb{R}$)

$$\left| \int f \, d\mu \right| = \left| \int f^+ \, d\mu - \int f^- \, d\mu \right| \underbrace{\le}_{\text{triangle inequality}} \int f^+ \, d\mu + \int f^- \, d\mu = \int f^+ + f^- \, d\mu = \int |f^+ + f^-| \, d\mu$$

($\mathbb{C}$) First, if $\int f \, d\mu = 0$ then nothing to prove, so assume $\int f \, d\mu \neq 0$ and define:

$$\alpha = \frac{\overline{\int f \, d\mu}}{|\int f \, d\mu|}$$

Observe:

$$\left| \int f \, d\mu \right| = \frac{(\int f \, d\mu) \left(\overline{\int f \, d\mu} \right)}{|\int f \, d\mu|} = \alpha \int f \, d\mu$$

and $|\int f \, d\mu| \in \mathbb{R}_+$, so:

$$\begin{aligned}
\left| \int f \, d\mu \right| = \alpha \int f \, d\mu &= \Re\left(\alpha \int f \, d\mu \right) \\
&= \Re\left(\int \alpha f \, d\mu \right) \\
&= \int \Re(\alpha f) \, d\mu && \text{(definition of complex integral)} \\
&= \left| \int \Re(\alpha f) \, d\mu \right| && \text{(as it's in } \mathbb{R}_+\text{)} \\
&\le \int |\Re(\alpha f)| \, d\mu && \text{(by above)} \\
&\le \int |\alpha f| \, d\mu \\
&= \int |\alpha| \cdot |f| \, d\mu \\
&= |\alpha| \int |f| \, d\mu \\
&= \left| \frac{\overline{\int f \, d\mu}}{|\int f \, d\mu|} \right| \int |f| \, d\mu \\
&= \int |f| \, d\mu
\end{aligned}$$

□

Proposition 24. *Let $f \in \mathcal{L}^1$:*

a) $\{x : f(x) \neq 0\}$ *is* σ*-finite and* $\{x : f(x) \in \{\pm\infty\}\}$ *has measure 0.*

b) Let $f, g \in \mathcal{L}^1$*, then:*

$$\int_E f\, d\mu = \int_E g\, d\mu \;\; \forall E \in \mathcal{M} \iff \int |f - g|\, d\mu = 0 \iff f = g \;\; a.e.$$

Definition 24. *A set is* **σ-finite** *if we can write it as a countable union of sets that have finite measure.*

Proof of proposition.

a) We only do the real case: w.l.o.g. assume f is non-negative (otherwise do same thing for f^+ and f^-):

$$f : X \to [0, \infty] \implies \{f \neq 0\} = \{f > 0\} = \bigcup_{n=1}^{\infty} \left\{ f > \frac{1}{n} \right\}$$

Need to check that $\mu(\{f > \frac{1}{n}\}) < \infty$. We use Chebychev's inequality:

$$\infty > \int f\, d\mu \geq \int f \chi_{\{f > \frac{1}{n}\}}\, d\mu \geq \int \frac{1}{n} \chi_{\{f > \frac{1}{n}\}}\, d\mu = \frac{1}{n} \mu\left(\left\{ f > \frac{1}{n} \right\} \right)$$

The second part is an exercise.

b) Assume $f = g$ a.e. Therefore:

$$f - g = 0 \text{ a.e.} \implies \underbrace{|f - g|}_{\text{non-negative}} = 0 \text{ a.e.} \implies \int |f - g|\, d\mu = 0$$

by earlier proof of nonnegative case. Now, assume $\int |f - g|\, d\mu = 0$. Then:

$$\left| \int_E f\, d\mu - \int_E g\, d\mu \right| = \left| \int_E f - g\, d\mu \right| \leq \int_E |f - g|\, d\mu \leq \int |f - g|\, d\mu = 0 \implies \int_E f\, d\mu = \int_E g\, d\mu \qquad (\forall E)$$

Now, assume $\int_E f\, d\mu = \int_E g\, d\mu$ $\forall E$ and $f, g : X \to \mathbb{R}$. Note:

$$\{f \neq g\} = \{f - g > 0\} \cup \{g - f > 0\}$$

So enough to show $\{f - g > 0\}$ has measure 0. Let:

$$E = \{f - g > 0\} \underbrace{=}_{\text{so } E \text{ is measurable}} (f - g)^{-1}((0, \infty])$$

For this E we have $\int_E f - g\, d\mu = 0$.

$$E = \{f - g > 0\} = \bigcup_{n=1}^{\infty} \left\{ f - g > \frac{1}{n} \right\}$$

For contradiction, assume $\mu(E) > 0$:

$$\mu\left(\left\{ f - g > \frac{1}{n} \right\} \right) \nearrow \mu(E) > 0$$

Which implies $\exists n$ such that $\mu(\{f - g > \frac{1}{n}\}) > 0$. Therefore:

$$\int_E f - g\, d\mu \leq \int_{\{f - g > \frac{1}{n}\}} f - g\, d\mu \geq \int_{\{f - g > \frac{1}{n}\}} \frac{1}{n}\, d\mu = \frac{1}{n} \mu\left(\left\{ f - g > \frac{1}{n} \right\} \right) > 0$$

So $f = g$ a.e.

□

We have $\mathcal{L}^1$, the space of measurable functions such that $\int |f| < \infty$. Try to define a norm:

$$\|f\| := \int |f| \, d\mu$$

Would hope for:

- $\|f\| \geq 0$, $\underbrace{\|f\| = 0 \iff f = 0}_{\text{the bit that fails}}$
- $\|\lambda f\| = |\lambda| \|f\|$ for $\lambda \in \mathbb{R}$
- $\|f + g\| \leq \|f\| + \|g\|$

How to fix the first property? We define an equivalence relation:

$$f \sim g \iff f = g \text{ a.e.}$$

Define:

$$L^1 = \mathcal{L}^1 / \sim \qquad \text{(Note: } [0] = [\chi_{\mathbb{Q}}]\text{)}$$

Now, L^1 is a metric space. We define:

$$[f] + [g] = [f + g]$$
$$\lambda [f] = [\lambda f]$$

In L^1, we can define:

$$\|[f]\| \underbrace{=}_{\text{well defined by a previous theorem}} \int |f| \, d\mu$$

Notation: we generally ignore the square brackets for practical purposes.

Theorem 9 (Dominated Convergence Theorem)**.** *Let* $\{f_n\}_{n=1}^{\infty}$, $f_n \in L^1$ *such that:*

a) $f_n \to f$ *a.e.*

b) $\exists g \in L^1$ *such that* $0 \leq |f_n| \leq g \ \forall n$. *Then:*

$$\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu$$

Proof. $|f_n| \leq g \implies g - f_n \geq 0$ and $f_n + g \geq 0$
Then apply Fatou's Lemma, which is:

$$\int \liminf_{n \to \infty} h_n \, d\mu \leq \liminf_{n \to \infty} \int h_n \, d\mu$$

for $h_n \geq 0$.

$$\implies \int \lim_{n \to \infty} (g - f_n) \, d\mu \leq \liminf_{n \to \infty} \int (g - f_n) \, d\mu \qquad \text{(as the limit exists)}$$

$$\text{and} \quad \int \lim_{n \to \infty} (g + f_n) \, d\mu \leq \liminf_{n \to \infty} \int (g + f_n) \, d\mu$$

$$\implies \int g - \lim_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \left(\int g \, d\mu - \int f_n \, d\mu \right)$$

$$\text{and} \quad \int g + \lim_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \left(\int g \, d\mu + \int f_n \, d\mu \right)$$

$$\implies \int g \, d\mu - \int f \, d\mu \leq \int g \, d\mu - \limsup_{n \to \infty} \int f_n \, d\mu \qquad (\liminf_{n \to \infty} (-a_n) = -\limsup_{n \to \infty} (a_n))$$

$$\int g \, d\mu + \int f \, d\mu \leq \int g \, d\mu + \liminf_{n \to \infty} \int f_n \, d\mu$$

$$\implies \limsup_{n \to \infty} \int f_n \, d\mu \leq \int f \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu$$

$$\implies \int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

□

Theorem 10. *$\{f_j\}_{j=1}^{\infty} \subset L^1$, assume $\sum_{n=1}^{\infty} \int |f_n| \, d\mu < \infty$. Then $\sum_{j=1}^{\infty} f_j$ converges a.e. Moreover:*

$$\sum_{j=1}^{\infty} f_j \in L^1 \qquad \text{(i.e. } \textstyle\int |\sum_{j=1}^{\infty} f_j| \, d\mu < \infty)$$

and:

$$\sum_{n=1}^{\infty} \int f_n \, d\mu = \int \sum_{n=1}^{\infty} f_n \, d\mu$$

Proof. By the monotone convergence theorem:

$$\infty > \sum_{k=1}^{\infty} \int |f_k| \, d\mu = \int \sum_{k=1}^{\infty} |f_k| \, d\mu$$

since:

$$\int \sum_{k=1}^{\infty} |f_k| \, d\mu < \infty \underbrace{\implies}_{\text{proposition}} \sum_{k=1}^{\infty} |f_k| < \infty \text{ a.e.}$$

Next, define $h_k = \sum_{n=1}^{k} f_n$. We want that $\lim_{k\to\infty} \int h_k \, d\mu = \int \lim_{k\to\infty} h_k \, d\mu$. Notice:

$$|h_k| = \left| \sum_{n=1}^{k} f_n \right| \le \sum_{n=1}^{k} |f_n| \le \underbrace{\sum_{n=1}^{\infty} |f_n|}_{\in L^1} < \infty$$

So, by the dominated convergence theorem:

$$\int \lim_{k\to\infty} h_k \, d\mu = \lim_{k\to\infty} \int h_k \, d\mu$$

□

Theorem 11. *L^1 is complete.*

Proof. Need to show that if $\{f_n\}_{n=1}^{\infty}$ is Cauchy then $\underbrace{\exists f \in L^1}_{1.}$ such that $\underbrace{f_n \to f}_{2.}$.

1. Assume w.l.o.g. that $f_1 \equiv 0$.

$$f_n - f_1 = f_n - f_{n-1} + f_{n-1} - \dots + f_2 - f_1$$

So:

$$f_n = \sum_{j=1}^{n-1} f_{j+1} - f_j$$

Assume (by taking a subsequence) that $\|f_{n+1} - f_n\| \le \frac{\varepsilon}{2^n}$. Denote $g_j := f_{j+1} - f_j$. Look at:

$$\sum_{j=1}^{\infty} \int |g_j| \, d\mu \le \sum_{j=1}^{\infty} \|f_{j+1} - f_j\| \le \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$$

By previous theorem:

$$\sum_{j=1}^{\infty} g_j < \infty \text{ a.e.} \implies \sum_{j=1}^{\infty} f_{j+1} - f_j < \infty \text{ a.e.}$$

Define: $f := \sum_{j=1}^{\infty} f_{j+1} - f_j$ and $f \in L^1$ by theorem.

2. (Want $\|f - f_n\| \to 0$ as $n \to \infty$)

$$\|f - f_n\| = \left\| \left(\sum_{j=1}^{\infty} f_{j+1} - f_j \right) - \left(\sum_{j=1}^{n-1} f_{j+1} - f_j \right) \right\| = \left\| \sum_{j=n}^{\infty} f_{j+1} - f_j \right\| \le \sum_{j=1}^{\infty} \|f_{j+1} - f_j\| \le \sum_{j=n}^{\infty} \frac{\varepsilon}{2^j} \le \frac{\varepsilon}{2^n}$$

□

Proposition 25 (Simple functions are dense in L^1)**.** *Let $f \in L^1(X, \mathcal{M}, \mu)$, then $\forall \varepsilon > 0$ $\exists \phi$ simple measurable such that:*

$$\|f - \phi\| = \int |f - \phi| \, d\mu < \varepsilon$$

If $X = \mathbb{R}$, μ the Lebesgue measure, then ϕ can be taken as $\phi = \sum_{j=1}^{N} a_j \chi_{E_j}$ where E_j are open intervals. Moreover, $\exists g$ continuous such that $\int |f - g| \, d\mu < \varepsilon$.

Proof. If $f : X \to [0, \infty]$ and $\exists \phi_n$ simple measurable such that:

$$0 \le \phi_1 \le \phi_2 \le \ldots \le \phi_n \le \ldots \le f$$

(Want to show that $\int |f - \phi_n| \, d\mu \to 0$) Let $h_n := f - \phi_n$. Then:

$$|h_n| \le |f| + |\phi_n| \le 2|f|$$

By DCT:

$$\lim_{n\to\infty} \int h_n \, d\mu = \int \lim_{n\to\infty} h_n \, d\mu = \int 0 \, d\mu = 0$$

In general case we know $f : X \to \mathbb{R}$ (or $\mathbb{C}$). We know:

$$0 \le |\phi_1| \le |\phi_2| \le \ldots \le |\phi_n| \le \ldots \le |f|$$

Let $h_n := |f - \phi_n|$, then $h_n \le |f| + |\phi_n| \le 2|f|$ and so $\lim_{n\to\infty} \int h_n \, d\mu = 0$ by DCT. Now, a sketch of the "moreover" statement. $X = \mathbb{R}$, μ = Lebesgue. $\forall \varepsilon$ $\exists \phi_n$ measurable such that $\int |f - \phi_n| \, d\mu < \frac{\varepsilon}{100}$. Suppose $\phi_n = \sum_{j=1}^{N} a_j \chi_{A_j}$. Since A_j is measurable we know $\exists \sigma_j$ open such that $\mu(\sigma_j \setminus A_j) \le \frac{\varepsilon}{10^{10^{10}}}$ and $\sigma_j \supset A_j$. Then $\widetilde{\phi_n} = \sum a_j \chi_{\sigma_j}$. $\int |\phi_n - \widetilde{\phi_n}|$ is very small. In $\mathbb{R}$ every open set is the union of open intervals. So:

$$\sigma_j = \dot{\bigcup_k} I_{j,k} \qquad \text{(for } I_{j,k} \text{ open interval)}$$

Since $\mu(\sigma_j) < \infty \implies \mu(I_j, k) \xrightarrow[k\to\infty]{0}$. Define $\widetilde{\sigma_j} = \bigcup_{k=1}^{M} I_{j,h}$ we can make $\mu(\sigma_j \setminus \widetilde{\sigma_j})$ very small. Define $\widetilde{\widetilde{\sigma_j}} = \sum a_j \chi_{a_j}$ (need to check $\int |f - \widetilde{\widetilde{\sigma_n}} \, d\mu < \varepsilon|$). $\int |f - \sigma_n + \sigma_n - \widetilde{\sigma_n} + \widetilde{\sigma_n} - \widetilde{\widetilde{\sigma_n}}| \, d\mu$ (then use triangle inequality). $\exists g$ continuous such that $\int |\chi_{[a,b]} - g| \, d\mu < \varepsilon$. □

Theorem 12. *Let $(X, \mathcal{M}, \mu)$, $f : X \times [a, b] \to \mathbb{R}$ (or $\mathbb{C}$), $x \in X$, $t \in [a, b]$. Assume $f(x, t)$ is integrable w.r.t. x $\forall t$. Define $F(t) = \int_X f(x, t) \, d\mu$.*

a) Assume $\exists g \in L^1(X, \mathcal{M}, \mu)$, $g \ge 0$ such that $|f(x, t)| \le g(x)$ and $\lim_{t\to t_0} f(x, t) = f(x, t_0)$. Then:

$$\lim_{t\to t_0} F(t) = F(t_0)$$

b) Suppose $\frac{\partial f}{\partial t}(x, t)$ exists and $\exists h \in L^1$ such that $|\frac{\partial f}{\partial t}(x, t)| \le h(x)$. Then:

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t) \, d\mu$$

Proof.

a) Define $f_n(x) = f(x, t_n)$ for some $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to t_0$. Then we have a sequence $\{f_n\}_{n=1}^{\infty}$ on X such that $f_n \in L^1$. Moreover, $|f_n(x)| \le g(x)$ so by DCT:

$$\int_X \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$$

Which implies:

$$F(t_0) := \int f(x, t_0) \, d\mu = \lim_{n\to\infty} \int f(x, t_n) \, d\mu = \lim_{n\to\infty} F(t_n)$$

b) Define $\frac{\partial f}{\partial t}(x, t_0) = \lim_{n\to\infty} h_n(x)$ where:

$$h_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}$$

and $t_n \to t_0$. $|h_n(x)| \le h(x)$, so apply DCT.

□

Theorem 13. *Let $f : [a,b] \to \mathbb{R}$.*

a) If f is Riemann integrable then it is Lebesgue integrable (and the two values are the same).

b) f is Riemann integrable iff $\{x : f(x) \text{ is discontinuous at } x\}$ has Lebesgue measure 0.

3.4 Different modes of convergence

Let $\{f_n\}_{n=1}^{\infty}$, $f_n : X \to \mathbb{R}$ (or $\mathbb{C}$)

1. **Uniform convergence:**

$$\forall \varepsilon > 0\ \exists N_\varepsilon \text{ s.t. } |f_n(x) - f_m(x)| < \varepsilon\ \forall m,n \ge N\ \forall x$$

2. **Pointwise convergence:**

$$\forall x\ \forall \varepsilon > 0\ \exists N_\varepsilon(x) \text{ s.t. } |f_n(x) - f_m(x)| < \varepsilon\ \forall n,m \ge N$$

3. **a.e. convergence:**

(Pointwise convergence except on a set of measure 0)

4. **L_1 convergence:**

$$\forall \varepsilon > 0\ \exists N \text{ s.t. } \int |f_n - f_m|\, d\mu < \varepsilon\ \forall n,m \ge N$$

5. **Convergence in measure:**

$$\forall \varepsilon > 0\ \mu(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}) \to 0 \text{ as } n,m \to \infty$$

Proposition 26. $L^1 \implies$ *measure.*

Proof. We have:

$$\int |f_n - f|\, d\mu \to 0$$

(Want that $\forall \varepsilon > 0$, $\mu(\{x : |f_n - f| > \varepsilon\}) \to 0$ as $n \to \infty$)

$$0 \leftarrow \int |f_n - f|\, d\mu \ge \int_{\{x:|f_n - f|>\varepsilon\}} |f_n - f|\, d\mu \ge \int_{\{x:|f_n-f|>\varepsilon\}} \varepsilon\, d\mu = \varepsilon\mu(\{x : |f_n - f| > \varepsilon\})$$

So $\mu(\{x : |f_n - f| > \varepsilon\}) \to 0$ as $n \to \infty$. □

Proposition 27. *Uniform* $\implies$ *measure*

Proof. $\forall \varepsilon > 0\ \exists N$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for $n \ge N\ \forall x$ (want $\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0$). For $n \ge N$, $\{x : |f_n(x) - f(x)| > \varepsilon\} = \varnothing$. □

Example 10 (Showing pointwise $\not\Longrightarrow$ measure)**.** *Consider $f_n(x) = \chi_{[n,n+1]}$. $f_n(x) = 0\ \forall n > x+1$, so $f_n \to 0$ pointwise but $\mu(\{x : |f_n(x) - 0| > \frac{1}{2}\}) = \mu([n, n+1]) = 1\ \forall n$.*

Example 11 (Showing uniform $\not\Longrightarrow L^1$). *Consider $f_n(x) = \frac{1}{n}\chi_{[0,n]}$. $\sup_{x\in\mathbb{R}} |\frac{1}{n}\chi_{[0,n]} - 0| = \frac{1}{n} \to 0$, therefore $f_n \rightrightarrows 0$ uniformly.*

$$\int \left|\frac{1}{n}\chi_{[0,n]} - 0\right| d\mu = n \cdot \frac{1}{n} = 1 \qquad (\forall n)$$

Example 12 (Showing $L^1 \not\Longrightarrow$ a.e.). *Construct a sequence as $f_1 = \chi_{(0,\frac{1}{2})}$, $f_2 = \chi_{(\frac{1}{2},\frac{1}{2}+\frac{1}{3})}$, continuing in this fashion until the positive end of the interval is greater than 1, such as in $f_3 = \chi_{(\frac{1}{2}+\frac{1}{3},\frac{1}{2}+\frac{1}{3}+\frac{1}{4})} = \chi_{(\frac{5}{6},\frac{13}{12})}$, so we define $f_4 = \chi_{(0,\frac{1}{5})}$, $f_5 = \chi_{(\frac{1}{5},\frac{1}{5}+\frac{1}{6})}$, and so on. We have:*

$$\int |f_n - 0|\, d\mu \leq \frac{1}{n+1} \to 0$$

So $\{f_n\}_{n=1}^{\infty}$ converges to 0 in L^1, but it doesnt convergence to 0 a.e. Let $x \in (0,1)$. For every n, $f_N(x) = 1$ for some $N > n$, so $f_n(x) \not\to 0$ $\forall x \in (0,1)$.

Theorem 14. *Let $\{f_n\}_{n=1}^{\infty} \subset L^1$. Assume f_n is Cauchy in measure. Then $\exists f$ such that f_n converges to f in measure. Furthermore, there is a subsequence $\{f_{n_j}\}_{n=1}^{\infty}$ such that $f_{n_j} \to f$ a.e. Moreover, if f_n converges in measure to g then $g = f$ a.e.*

Proof. We know $\forall \varepsilon > 0$ $\mu(\{x : |f_n(x) - f_m(x)| > \varepsilon\}) \to 0$ as $m, n \to \infty$. i.e. $\forall \varepsilon > 0$ $\forall \delta > 0$ $\exists N$ s.t. $\mu(\{x : |f_n(x) - f_m(x)| > \varepsilon\}) < \delta$ $\forall m, n > N$. Choose $\varepsilon = \frac{1}{2^j}$, $\delta = \frac{1}{2^j}$. Then $\exists N_j$ s.t. $\mu(\{x : |f_n(x) - f_m(x)| > \frac{1}{2^j}\}) < \frac{1}{2^j}$ for all $m, n \geq N_j$. Define $g_j := f_{N_j}$ (want to show $g_j(x)$ converges to something). So:

$$\underbrace{\mu\left(\left\{x : |g_j(x) - g_{j+1}(x)| > \frac{1}{2^j}\right\}\right)}_{E_j} < \frac{1}{2^j}$$

Define:

$$F_k = \bigcup_{j=k}^{\infty} E_j$$

F_k points where we shouldn't hope for convergence.

$$\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j) \leq \sum_{j=k}^{\infty} \frac{1}{2^j} = \frac{2}{2^k}$$

What happens outside F_k? Take $x \notin F_k$, look at $|g_j(x) - g_i(x)|$. w.l.o.g. assume $j \geq i$:

$$\begin{aligned}|g_j(x) - g_i(x)| &= |g_j(x) - g_{j-1}(x) + g_{j+1}(x) - ... + g_{i+1}(x) - g_i(x)| \\ &\leq \sum_{l=i}^{j-1} |g_{l+1}(x) - g_l(x)|\end{aligned}$$

Remember $x \notin F_k$ so take $i, j \geq k$, then:

$$|g_j(x) - g_i(x)| \leq \sum_{l=i}^{j-1} \frac{1}{2^l} \leq \sum_{l=i}^{\infty} \frac{1}{2^l} = \frac{2}{2^i} \leq \frac{2}{2^k}$$

This means $\{g_j(x)\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Define $f(x)$ to be the limit of $g_n(x)$ as $n \to \infty$ for $x \notin F_k$. We have now defined $f(x)$ for $x \in F_k^c$ for every k. So we have $f(x)$ for $x \in \bigcup_{k=1}^{\infty}(F_k^c)$. We now need only define f for $x \in (\bigcup_{k=1}^{\infty}(F_k^c))^c$, but $(\bigcup_{k=1}^{\infty}(F_k^c))^c = \bigcap_{k=1}^{\infty} F_k$. Notice that:

$$\mu\left(\bigcap_{k=1}^{\infty} F_k\right) \leq \mu(F_k) \leq \frac{2}{2^k} \implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = 0$$

So define f to be anything you like in $\bigcap_{k=1}^{\infty} F_k$. Thus we have shown that there is a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that $f_{n_j} \to f$ a.e. Next, want to show $f_n \to f$ in measure. We know $|g_j(x) - g_i(x)| \leq \frac{2}{2^i}$ for $x \notin F_k$, $j \geq i \geq k$. Taking limits as $j \to \infty$ (they exist by above):

$$\lim_{j\to\infty} |g_j(x) - g_i(x)| \leq \frac{2}{2^i} \qquad (\text{for } x \notin F_k,\ i \geq k)$$

$$|f(x) - g_i(x)| \leq \frac{2}{2^i} \qquad (\text{for } x \notin F_n,\ i \geq k)$$

We know $\mu(\{x : |g_j(x) - f(x)| \geq \frac{2}{2^j}\}) \leq \mu(F_k)$. Given $\varepsilon > 0$ choose j such that $\frac{2}{2^j} < \varepsilon$:

$$\mu(\{x : |g_j(x) - f(x)| > \varepsilon\}) \leq \mu\left(\left\{x : |g_j(x) - f(x)| > \frac{2}{2^j}\right\}\right) \leq \mu(F_k)$$

Letting $j = k$ gives:

$$\mu(\{x : |g_j(x) - f(x)| > \varepsilon\}) \leq \mu(F_j) \leq \frac{2}{2^j} \to 0 \qquad (\text{as } j \to \infty)$$

So far we've *only* shown g_i converges to f in measure, not that f_n converges to f in measure. So:

$$\begin{aligned} \{x : |f_n(x) - f(x)| > \varepsilon\} &= \{x : |f_n(x) - g_i(x) + g_i(x) - f(x)| > \varepsilon\} \\ &\subset \left(\left\{x : |f_n(x) - g_i(x)| > \frac{\varepsilon}{2}\right\} \cup \left\{x : |g_i(x) - f(x)| > \frac{\varepsilon}{2}\right\}\right) \\ \implies \quad \mu(\{x : |f(x)_n - f(x)| > \varepsilon\}) &\leq \underbrace{\mu\left(\left\{x : |f_n(x) - f_{N_i}(x)| > \frac{\varepsilon}{2}\right\}\right)}_{\substack{\text{Theorem hypothesis that} \\ f_n \text{ Cauchy in measure}}} + \underbrace{\mu\left(\left\{x : |f_{N_i}(x) - f(x)| > \frac{\varepsilon}{2}\right\}\right)}_{\to 0 \text{ as } i \to \infty \text{ as shown}} \end{aligned}$$

Finally, uniqueness, let there be two, f and g, then:

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0 \qquad (\text{as } n \to \infty)$$
$$\mu(\{x : |g_n(x) - g(x)| > \varepsilon\}) \to 0 \qquad (\text{as } n \to \infty)$$

But:

$$\begin{aligned} \{x : |f - g| > \varepsilon\} &= \{x : |f - f_n + f_n - g| > \varepsilon\} \\ &\subseteq \left(\left\{x : |f - f_n| > \frac{\varepsilon}{2}\right\} \cup \left\{x : |f_n - g| > \frac{\varepsilon}{2}\right\}\right) \\ \implies \quad \mu(\{x : |f - g| > \varepsilon\}) &\leq \mu\left(\left\{x : |f - f_n| > \frac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{x : |f_n - g| > \frac{\varepsilon}{2}\right\}\right) \\ \text{LHS} &\leq \lim_{n \to \infty} \text{RHS} = 0 \\ \implies \quad \mu(\{x : |f - g| > \varepsilon\}) &= 0 \end{aligned}$$

So:

$$\{x : f \neq g\} = \bigcup_{n=1}^{\infty} \left\{x : |f - g| > \frac{1}{2^n}\right\}$$
$$\mu(\{x : f \neq g\}) \leq \sum_{n=1}^{\infty} \mu\left(\left\{x : |f - g| > \frac{1}{2^n}\right\}\right) = \sum_{n=1}^{\infty} 0 = 0 \implies f = g \text{ a.e.}$$

□

Corollary 4. *If $\{f_n\}_{n=1}^{\infty} \subset L^1$ is Cauchy in L^1, then there is a subsequence $f_{n_j} \to f$ a.e.*

Theorem 15 (Egorov)**.** *Let $f : X \to \mathbb{R}$, $\mu(X) < \infty$, $f_n : X \to \mathbb{R}$ and $f_n \to f$ a.e. Then $\forall \varepsilon > 0$ there is a set E with $\mu(E) < \varepsilon$ such that $f_n \rightrightarrows f$ on E^c.*

Proof. w.l.o.g. $f_n \to f$ $\forall x$ (add to E the set where you don't converge). Define:

$$E_n(k) = \bigcup_{m=n}^{\infty} \left\{x : |f_m(x) - f(x)| > \frac{1}{k}\right\}$$

Notice $E_n(k) \supseteq E_{n+1}(k)$ with k fixed:

$$\lim_{n \to \infty} E_n(k) = \bigcap_{n=1}^{\infty} E_n(k) = \varnothing$$

and:

$$\mu(E_1(k)) \leq \mu(X) < \infty$$

So:

$$\underbrace{\mu\left(\bigcap_{n=1}^{\infty} E_n(k)\right)}_{=\mu(\varnothing)=0} = \lim_{n\to\infty} \mu(E_n(k))$$

So $\forall k$, given $\varepsilon > 0$ $\exists n(k)$ (i.e. n depending on k) such that $\mu(E_{n(k)}(k)) < \frac{\varepsilon}{2^k}$. Define:

$$E = \bigcup_{k=1}^{\infty} E_{n(k)}(k)$$

Then:

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_{n(k)}(k)) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

(Need to show $f_n \rightrightarrows f$ on E^c)

$$E^c = \bigcap_{k=1}^{\infty} (E_{n(k)}(k))^c$$

So:

$$x \in E^c \implies x \in (E_{n(k)}(k))^c \qquad (\forall k \in \mathbb{N})$$

But:

$$\begin{aligned}(E_{n(k)}(k))^c &= \bigcap_{m=n(k)}^{\infty} \left(\left\{x : |f_m(x) - f(x)| > \frac{1}{k}\right\}^c\right) \\ &= \bigcap_{m=n(k)}^{\infty} \left\{x : |f_m(x) - f(x)| \leq \frac{1}{k}\right\}\end{aligned}$$

So:

$$x \in E^c \implies |f_m(x) - f(x)| \leq \frac{1}{k} \qquad (\forall k, \text{ for } m \text{ large enough})$$

Which implies uniform convergence. □

3.5 Product measures

Reminder of the product σ-algebra: take $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$. We defined the product σ-algebra in terms of:

$$\left\{\prod_{\alpha}^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha\right\}$$

We denoted this as $\otimes \mathcal{M}$. When you have a finite (or countable) product, that is the same as the one generated by:

$$\left\{\prod E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}$$

Definition 25. *A* **rectangle** *is any set of the form $A \times B$ for $A \in \mathcal{M}, B \in \mathcal{N}$.*

Observations:

- $(A \times B) \cap (E \times F)$ is a rectangle as $(A \times B) \cap (E \times F) = (A \cap E) \times (B \cap F)$.
- $(A \times B)^c = (X \times B^c) \cup (A^c \times B)$

Claim. The collection of finite disjoint unions of rectangles is an algebra.
Consider the rectangle $A \times B$. Assume $A \times B$ can be written as $\dot\bigcup_{i=1}^{\infty} A_i \times B_i$ with $A_i \times B_i$ rectangles. So:

$$\chi_A(x)\chi_B(y) = \chi_{A\times B}(x,y) = \sum_{i=1}^{\infty} \chi_{A_i \times B_i}(x,y) = \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y)$$

Then:

$$\int \chi_A(x)\chi_B(y)\,d\mu = \int \chi_{A\times B}(x,y)\,d\mu = \int \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y)\,d\mu \underbrace{=}_{\text{By MCT}} \sum_{i=1}^{\infty} \int \chi_{A_i}(x)\chi_{B_i}(y)\,d\mu = \sum_{i=1}^{\infty} \mu(A_i)\chi_{B_i}(y)$$

Now, we integrate with respect to y:

$$\mu(A)\nu(B) = \int \mu(A)\chi_B(y)\,d\nu = \iint \chi_{A\times B}(x,y)\,d\mu\,d\nu \underbrace{=}_{\text{MCT}} \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i)$$

If we could define a measure π on $(X \times Y, \mathcal{M} \otimes \mathcal{N})$, we would expect:

$$\pi(A \times B) = \iint_{X\times Y} \chi_{A\times B}\,d\pi$$

We find $\pi(A \times B)$ should be $\mu(A)\nu(B)$ or $\sum_{i=1}^{\infty} \mu(A_i)\nu(B_i)$. Thus:

$$\pi(A \times B) = \mu(A)\nu(B) = \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) = \sum_{i=1}^{\infty} \pi(A_i \times B_i)$$

Construction: define $\pi(A \times B) \coloneqq \mu(A)\nu(B)$ as it has the property $\pi\left(\dot\bigcup_{i=1}^{\infty}(A_i \times B_i)\right) = \sum_{i=1}^{\infty} \pi(A_i \times B_i)$. Now we want to define an outer measure. Given any set $W \in \mathcal{M} \otimes \mathcal{N}$, define:

$$\pi^*(W) = \inf\left\{\sum_{i=1}^{\infty} \pi(F_i \times G_i) : W \subset \bigcup_{i=1}^{\infty}(F_i \times G_i), F_i \in \mathcal{M}, G_i \in \mathcal{N}\right\}$$

Definition 26. *A set A is measurable if:*

$$\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c) \qquad (\forall E)$$

Applying Caratheodory's theorem we get that the set of measurable sets is a σ-algebra (in fact it is the same as $\mathcal{M} \otimes \mathcal{N}$).

Main point: π is an extension of $\pi(A \times B) = \mu(A)\nu(B)$ when $A \times B$ is a rectangle.

Note:

- If μ and ν are σ-finite then $\infty \cdot 0 = 0$.
- Else we cannot say anything.

Example 13. *Consider* $\mathbb{R} \times \{1\}$.

$$\pi(\mathbb{R} \times \{1\}) = \mu(\mathbb{R})\nu(\{1\}) = \infty \cdot 0$$

But:

$$\mathbb{R} \times \{1\} = \bigcup_{n=1}^{\infty}([-n,n] \times \{1\})$$

So:

$$\pi(\mathbb{R} \times \{1\}) = \pi\left(\bigcup_{n=1}^{\infty}([-n,n] \times \{1\})\right) \underbrace{=}_{\textit{continuity}} \lim_{n\to\infty} \pi([-n,n] \times \{1\}) = \lim_{n\to\infty} 0 = 0$$

Let $E \subset X \times Y$.

$$E_x = \{y \in Y : (x,y) \in E\}$$
$$E^y = \{x \in X : (x,y) \in E\}$$

$f : X \times Y \to \mathbb{R}$ *(or* $\mathbb{C}$*).*

- $f_x(y)$ *fixes* x*, function on* y*.*
- $f^y(x)$ *fixes* y*, function on* x*.*

Proposition 28.

a) Let $E \subset X \times Y$*,* $E \in \mathcal{M} \otimes \mathcal{N}$*. Then* E_x *is measurable w.r.t.* ν *and* E^y *is measurable w.r.t.* μ*.*

b) Let $f : X \times Y \to \mathbb{R}$ *(or* $\mathbb{C}$*) be a map that is* $\mathcal{M} \otimes \mathcal{N}$ *measurable. Then* $f_x(y)$ *is measurable w.r.t.* $\mathcal{N}$ *and* $f_y(x)$ *is measurable w.r.t.* $\mathcal{M}$*.*

Proof.

a) Remember, $\mathcal{M} \otimes \mathcal{N}$ is σ-algebra generated by $A \times B$ s.t. $A \in \mathcal{M}$ and $B \in \mathcal{N}$. So it is smallest σ-algebra containing all $A \times B$.

$$\{E : E \subset X \times Y \text{ and } E_x \text{ is measurable w.r.t. } \nu\} \supset \mathcal{M} \otimes \mathcal{N}$$

Want to show two things for this set:

i) It contains all rectangles $A \times B$:
$E = A \times B$ rectangle. Then:

$$E_x = \begin{cases} B & \text{if } x \in A \\ \varnothing & \text{if } x \notin A \end{cases}$$

ii) It's a σ-algebra:
We need to show that it is closed under countable unions and closure under complements. Let $\{E_i\}_{i=1}^{\infty} \subset R$

- $$\left(\bigcup_{i=1}^{\infty} E_i\right)_x = \bigcup_{i=1}^{\infty} (E_{i_x}) \in \mathcal{N}$$
- $$(E^c)_x = (E_x)^c \in \mathcal{N}$$

b) Need f_x measurable w.r.t. $\mathcal{N}$, need:

$$(f_x)^{-1}(\sigma) \in \mathcal{N} \qquad (\forall \sigma \text{ Borel})$$

We know:

$$f^{-1}(\sigma) \in \mathcal{M} \otimes \mathcal{N} \qquad (\forall \sigma \text{ Borel})$$

Claim. $(f^{-1}(\sigma))_x = (f_x)^{-1}(\sigma)$

□

Theorem 16. *Let* $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ *be* σ*-finite measurable spaces. For* $E \in \mathcal{M} \otimes \mathcal{N}$ *we define:*

$$x \mapsto \nu(E_x)$$
$$y \mapsto \mu(E^y)$$

The the two functions are measurable (w.r.t. appropriate measure), and:

$$\mu \times \nu(E) = \int \nu(E_x)\, d\mu = \int \mu(E^y)\, d\nu$$

So:

$$\begin{aligned} \mu \times \nu(E) &= \iint_{X \times Y} \chi_E(x, y)\, d\mu \times \nu && \text{(was true before)} \\ &= \int_X \left(\int_Y \chi_E(x, y)\, d\nu\right) d\mu = \int_Y \left(\int_X \chi_E(x, y)\, d\mu\right) d\nu && \text{(the theorem)} \end{aligned}$$

Theorem 17 (Fubini-Tonelli). *Let $(X,\mathcal{M},\mu),(Y,\mathcal{N},\nu)$ be σ-finite measure spaces.*

a) (Tonelli) Let f be a non-negative measurable function in $X\times Y$. Define:

$$g(x) \coloneqq \int f_x \, d\nu$$
$$h(y) \coloneqq \int f^y \, d\mu$$

Then g and h are measurable functions (w.r.t. appropriate σ-algebra). Moreover:

$$\int f \, d\mu\times\nu = \int g \, d\mu = \int h \, d\nu$$

b) (Fubini) Let $f \in L^1(\mu\times\nu)$ (i.e. $\int |f| \, d\mu\times\nu < \infty$). Then:

$$f_x \in L^1(\nu) \qquad \text{(for almost every } x \in X)$$
$$f_y \in L^1(\mu) \qquad \text{(for almost every } y \in Y)$$

Then if:

$$g(x) \coloneqq \int f_x \, d\nu$$
$$h(y) \coloneqq \int f^y \, d\mu$$

We have $g \in L^1(\mu)$ and $h \in L^1(\nu)$. Moreover:

$$\int f \, d\mu\times\nu = \int g \, d\mu = \int h \, d\nu$$

Note that part a) (Tonelli) does not require $\int f \, d\mu\times\nu < \infty$. In practical terms, given $f : X\times Y \to \mathbb{R}$. We look at $|f| : X\times Y \to [0,\infty)$, then $|f|$ satisfied all assumptions of Tonelli's theorem. This will tell us if $f \in L^1$, if it is then we can apply Fubini (if not then we know nothing...).

Proof.

a) Claims:

  i) The preceding theorem is a special case of Fubini-Tonelli, when $f(x,y) = \chi_E(x,y)$
  ii) By linearity Fubini-Tonelli is true for linear combinations of indicator functions.
  iii) We can finish proof using fact that we can approximate measurable functions with an increasing sequence of linear combinations of characteristic functions.
  iii) Given $f \geq 0$, we construct a sequence of simple functions $\{\phi_n\}_{j=1}^{\infty}$, i.e. each a linear combination of characteristic functions:

$$0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq f$$

And:

$$\int \phi_n \, d\mu\times\nu = \int \left(\int (\phi_n)_x \, d\nu \right) d\mu = \int \left(\int (\phi_n)^y \, d\mu \right) d\nu$$

Now:

$$g_n(x) \coloneqq \int (\phi_n)_x(y) \, d\nu$$

g_n is monotone increasing as $\phi_k \leq \phi_{k+1}$:

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \int (\phi_n)_x \, d\nu = \int \lim_{n\to\infty} \phi_n(x,y) \, d\nu = g(x)$$

So:

$$\begin{aligned}
\int g\,d\mu &= \int \lim_{n\to\infty} g_n\,d\mu \\
&= \lim_{n\to\infty} \int g_n\,d\mu \\
&= \lim_{n\to\infty} \left(\int \left(\int (\phi_n)_x(y)\,d\nu \right) d\mu \right) \\
&= \lim_{n\to\infty} \left(\int \left(\int \phi_n(x,y)\,d\mu \right) d\nu \right) \\
&= \int \left(\lim_{n\to\infty} \int \phi_n(x,y)\,d\mu \right) d\nu \\
&= \int \left(\int f(x,y)\,d\mu \right) d\nu
\end{aligned}$$

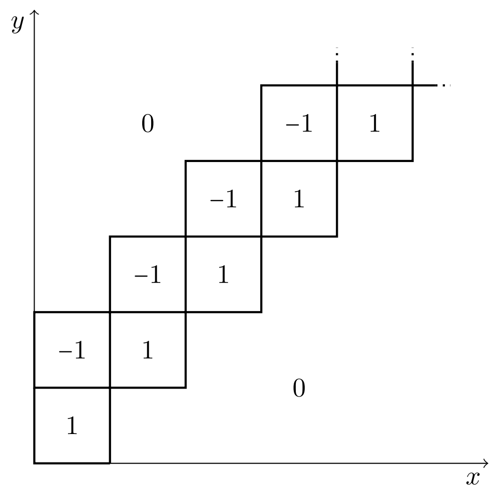
b) We are assuming $\int |f|\,d\mu\times\nu < \infty$, which implies $\int f^+\,d\mu\times\nu < \infty$ and $\int f^-\,d\mu\times\nu < \infty$, with:

$$\int f\,d\mu\times\nu = \int f^+\,d\mu\times\nu - \int f^-\,d\mu\times\nu$$

Apply part a) to each of the two integrals and then recombine.

□

Example 14.



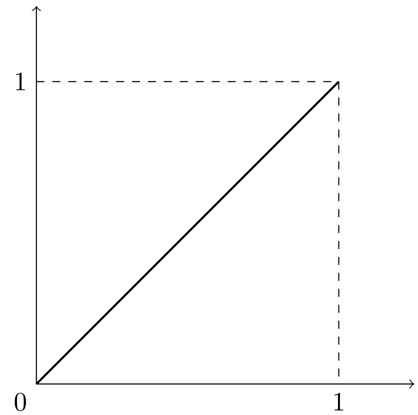
$$\int \left(\int f(x,y)\,dx \right) dy = 1$$
$$\int \left(\int f(x,y)\,dy \right) dx = 0$$

The reason they differ is that:

$$\iint |f(x,y)|\,dx\,dy = \infty$$

Example 15. *Let:*

- $\mu(A) = \#$ of points in A
- $X = [0,1] = Y$
- $\mathcal{M} = \mathcal{P}(X)$
- μ = counting measure
- $\mathcal{N}$ = Lebesgue
- ν = Lebesgue measure
- $D = \{(x,x) : x \in [0,1]\}$



Consider:

$$\iint \chi_D(x,y)\, d\mu \times \nu$$

We have:

$$\int \left(\int \chi_D(x,y)\, d\mu \right) d\nu = \int_{[0,1]} 1\, d\nu = 1$$
$$\int \left(\int \chi_D(x,y)\, d\nu \right) d\mu = \int_{[0,1]} 0\, d\mu = 0$$

Claim.

$$\iint \chi_D(x,y)\, d\mu \times \nu = \mu \times \nu(D) = \infty$$

4 Signed Measures

Let $(X, \mathcal{M})$ be a measurable space. Then we say ν is a signed measure if it satisfies $\nu : \mathcal{M} \to [-\infty, \infty]$ with:

1. $\nu(\varnothing) = 0$
2. ν takes at most one of $\pm\infty$.
3. $\{E_j\}_{j=1}^{\infty}$, with $E_j \in \mathcal{M}$ pairwise disjoint then:

$$\nu\left(\dot{\bigcup_{n=1}^{\infty}} E_n \right) = \sum_{n=1}^{\infty} \nu(E_n)$$

   with the series converging absolutely for $\sum_{n=1}^{\infty} \nu(E_n) < \infty$.

Example 16.

1. *Take two measures on the same measurable space $(X, \mathcal{M})$. Say μ_1, μ_2 with $\mu_1(X), \mu_2(X) < \infty$. Define:*

$$\nu(A) := \mu_1(A) - \mu_2(A)$$

2. *Take μ a (positive) measure. $f : X \to [-\infty, \infty]$ measurable with $\int |f|\, d\mu < \infty$. Define:*

$$\nu(E) := \int_E f\, d\mu$$

We know that if μ is a positive measure and $f \geq 0$ then the map $A \mapsto \int_A f\, d\mu$ is a measure. Write:

$$\nu(E) = \int_E f\, d\mu = \int_E f^+\, d\mu - \int_E f^-\, d\mu$$

Proposition 29. *Let ν be a signed measure on $(X\mathcal{M})$.*

i) If $\{E_j\} \subset \mathcal{M}$, $E_j \subset E_{j+1}$, then:

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \nu(E_j)$$

ii) If $\{f_j\} \subset \mathcal{M}$, $F_j \supset F_{j+1}$ and $|\nu(F_1)| < \infty$, then:

$$\nu\left(\bigcap_{j=1}^{\infty} F_j\right) = \lim_{j\to\infty} \nu(F_j)$$

Definition 27. *Let $(X, \mathcal{M})$, and let ν be a signed measure.*

1. *We say $E \in \mathcal{M}$ is a* **positive set** *iff $\nu(F) \geq 0$ $\forall F \in \mathcal{M}$ with $F \subset E$.*
2. *We say $E \in \mathcal{M}$ is a* **negative set** *iff $\nu(F) \leq 0$ $\forall F \in \mathcal{M}$ with $F \subset E$.*
3. *We say $E \in \mathcal{M}$ is* **null** *if E is positive and negative.*

Example 17. *$\nu(E) = \int_E f\, d\mu$ for some $f \in L^1$, μ positive measure. Take $f(x) = x$, $E \subset [-1,1]$, μ Lebesgue.*

$$\nu(E) = \int_E x\, dx$$
$$\nu([-1,1]) = \int_{-1}^{1} x\, dx = 0$$

But:

$$[0,1] \subset [-1,1] \text{ and } \nu([0,1]) = \frac{1}{2}$$
$$[-1,0] \subset [-1,1] \text{ and } \nu([-1,0]) = -\frac{1}{2}$$

So E is neither positive or negative.

Lemma 3.

1. *Any subset (that is measurable) of a positive set is positive.*
2. *Any subset (that is measurable) of a negative set is negative.*

Theorem 18 (Hahn decomposition theorem)**.** *Let ν be a signed measure on $(X, \mathcal{M})$-measurable space. Then $\exists P \in \mathcal{M}$, a positive set and $N \in \mathcal{M}$, a negative set such that $X = P \uplus N$. If P', N' are another such pair, then $P \triangle P'$ and $N \triangle N'$ are null.*

Notation: $P \triangle P'$ is the **symmetric difference**.

$$P \triangle P' = (P \cup P') \setminus (P \cap P') = (P \setminus P') \cup (P' \setminus P)$$

Proof. w.l.o.g. ν does not attain $+\infty$. Define:

$$M = \sup_{E\in\mathcal{M}} \underbrace{\{\nu(E) : E \text{ positive}\}}_{\text{not empty as it contains } \varnothing}$$

Which implies $\exists\{P_j\}_{j=1}^{\infty} \subset \mathcal{M}$ such that P_j positive and $\nu(P_j) \nearrow \mathcal{M}$.
Claim. $P := \bigcup_{j=1}^{\infty} P_j$ is positive and $N := X \setminus P$ is negative.
Let $E \subset P$, then:

$$E = E \cap P = E \cap \left(\bigcup_{j=1}^{\infty} P_j\right) = \bigcup_{j=1}^{\infty}(E \cap P_j) \qquad \text{(and } \nu(E \cap P_j) \geq 0\ \forall j)$$

So P is positive.

Observe that N does not contain any positive sets of positive measure. Otherwise, take $E \subset N$ with $\nu(E) > 0$. Then $\nu(P \cup E) = \nu(P) + \nu(E) > M$.

To show N is negative, go by contradiction. Assume N is not negative, i.e $\exists A \subset N$ such that $\nu(A) > 0$. Then, as A cannot be positive, $\exists C \subset A$ such that $\nu(C) < 0$. So take $B = A \setminus C$, then as:

$$\nu(A) = \nu(C) + \nu(A \setminus C)$$

We have $\nu(B) > \nu(A)$. We now construct a sequence $\{A_j\}_{j=1}^{\infty} \subset N$ and sequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$. Let n_1 be the smallest natural number such that $\exists B \subset N$ with $\nu(B) > \frac{1}{n_1}$. Choose A_1 to be one such set B. Let n_j be the smallest natural number such that $\exists B \subset A_{j-1}$ with $\nu(B) > \nu(A_{j-1}) + \frac{1}{n_j}$. Choose A_j to be one such set B. Define $A = \bigcap_{j=1}^{\infty} A_j$, then $\nu(A) = \lim_{j\to\infty} \nu(A_j) \geq \sum_{j=1}^{\infty} \frac{1}{n_j} \implies n_j \to \infty$ as $j \to \infty$ (as ν does not attain $+\infty$). So $\nu(A) > 0$ but $\exists B$ such that $\nu(B) > \nu(A) + \frac{1}{n_*}$ for some n_*. Notice, as $n_j \to \infty$, at some point $n_j > n_*$. Once $n_j > n_*$ we have a contradiction as n_j is, by definition, the smallest natural number such that $\exists B$ with:

$$\nu(B) > \nu(A_{j-1}) + \frac{1}{n}$$

So N is negative.
Finally, need to show uniqueness of decomposition (i.e. if P', N' is another such pair then $P \triangle P'$ and $N \triangle N'$ are null; we will do this by showing they are both positive and negative).
Need $P \setminus P'$ and $P' \setminus P$ to be both positive and negative.

$$\begin{aligned} P \setminus P' \subset P &\implies \text{positive} \\ P' \setminus P \subset P' &\implies \text{positive} \\ P \setminus P' \subset N' &\implies \text{negative} \\ P' \setminus P \subset N &\implies \text{negative} \end{aligned}$$

□

Notation: any decomposition $X = P \cup N$ with P positive and N negative is called a **Hahn decomposition**.

Definition 28. *Let μ and ν be two signed measures on a non-empty measurable space $(X, \mathcal{M})$. We say μ is* **mutually singular** *w.r.t. ν if $\exists F, E \in \mathcal{M}$ such that:*

$$X = F \cup E, E \cap F = \varnothing$$

with E null for μ and F null for ν.

Theorem 19 (Jordan decomposition)**.** *Given a signed measure ν on $(X, \mathcal{M})$, there exists two unique positive measures μ^+, μ^- that are mutually singular and satisfy:*

$$\nu = \mu^+ - \mu^-$$

Proof.

$$\begin{aligned} \mu^+(E) &= \nu(E \cap P) \\ \mu^-(E) &= -\nu(E \cap N) \end{aligned}$$

where P, N are Hahn decompositions of X, which implies μ^+ and μ^- are positive.

$$\nu(E) = \nu(E \cap (P \cup N)) = \nu(E \cap P) + \nu(E \cap N) = \mu^+(E) - \mu^-(E)$$

μ^+ and μ^- are mutually singular as $X = P \uplus N$. Let $E \subset N$. Then:

$$\mu^+(E) = \nu(\underbrace{E \cap P}_{\varnothing}) = 0$$

For uniqueness let:

$$\nu = \mu^+ - \mu^-$$
$$\nu = \nu^+ - \nu^-$$

with $\nu^+ \neq \mu^+$ and $\nu^- \neq \mu^-$. The measures ν^+ and ν^- must then generate another Hahn decomposition as ν^+ and ν^- are mutually singular. Therefore $\exists E, F$ such that $X = E \cup F$, $E \cap F = \varnothing$ with E null for ν^-, F null for ν^+. Now:

$$\mu^+(A) = \mu^+(A \cap P) = \nu(A \cap P) = \nu(A \cap E) \underbrace{=}_{\nu(A\cap E)=\nu^+(A\cap E)-\nu^-(A\cap E)} \nu^+(A \cap E) \overbrace{=}^{\nu^+(A)=\nu^+(A\cap(E\uplus F))=\nu^+(A\cap E)+\nu^+(A\cap F)} \nu^+(A) \qquad (\forall A)$$

Also, $\nu(A \cap P) = \nu(A \cap E)$ as:

$$\begin{aligned}
\nu(A \cap P) &= \nu(A \cap P \cap (E \uplus F)) \\
&= \nu(A \cap P \cap E) + \cancel{\nu(A \cap P \cap F)} && \text{(as } A \cap P \cap F \subset P \text{ positive)} \\
\nu(A \cap E) &= \nu(A \cap E \cap (P \uplus N)) \\
&= \nu(A \cap P \cap E) + \cancel{\nu(A \cap E \cap N)} && \text{(as } A \cap P \cap F \subset F \text{ negative)}
\end{aligned}$$

□

Observation: $(X, \mathcal{M}, \nu)$ with ν signed.

- In the Hahn decomposition, P, N are not necessarily unique.
- Jordan decomposition, ν^+, ν^- *are* unique.

Definition 29. $|\nu| := \nu^+ + \nu^-$ *is the* **total variation**.

$\mu \perp \nu$ whenever μ and ν mutually singular.

Exercise (highly examinable):

$$\nu \perp \mu \iff |\nu| \perp \mu \iff \nu^+ \perp \mu \text{ and } \nu^- \perp \mu$$

Recall:

$$\nu(E) := \int_E f \, d\mu \qquad \text{(for } f \in L^1\text{, } \mu \text{ positive)}$$

Proposition 30. *Given a signed measure ν, we have:*

$$\nu(E) = \int_E f \, d\mu$$

Where $f = \chi_P - \chi_N$ for P, N from Hahn decomposition, and $\mu = |\nu|$.

Proof.

$$\begin{aligned}
\int_E f \, d\mu &= \int_E \chi_P - \chi_N \, d\mu \\
&= \int (\chi_P - \chi_N)\chi_E \, d|\nu| \\
&= \int \chi_{P \cap E} - \chi_{N \cap E} \, d(\nu^+ + \nu^-) \\
&= \nu^+(P \cap E) - \nu^+(N \cap E) + \nu^-(P \cap E) - \nu^-(N \cap E) \\
&= \nu^+(P \cap E) - \nu^-(N \cap E) \\
&= \nu^+(E) - \nu^-(E) \\
&= \nu(E)
\end{aligned}$$

□

How to integrate w.r.t. signed measures?

$$\nu = \nu^+ - \nu^- \qquad \text{(unique by Jordan)}$$
$$\int f\,d\nu := \int f\,d\nu^+ - \int f\,d\nu^- \qquad \text{(whenever this is not } \infty - \infty\text{)}$$

Definition 30. *Let $f : I \to \mathbb{R}$, f is* **absolutely continuous** *if $\forall \varepsilon > 0\ \exists \delta$ such that whenever a finite sequence of pairwise disjoint subintervals $(a_k, b_k) \subset I$ satisfies $\sum_k |b_k - a_k| < \delta$, we have $\sum_k |f(b_k) - f(a_k)| < \varepsilon$.*

Definition 31. *μ-positive measure, ν-signed measure. ν is* **absolutely continuous** *w.r.t. μ if:*

$$\mu(E) = 0 \implies \nu(E) = 0$$

We write $\nu << \mu$

Exercises:

1. $$\nu << \mu \iff \nu^+ << \mu \text{ and } \nu^- << \mu$$

2. $$\nu \perp \mu \text{ and } \nu << \mu \implies \nu(A) = 0 \qquad (\forall A)$$

Theorem 20. *ν-signed measure, μ-positive measure. Then $\nu << \mu$ iff $\forall \varepsilon > 0\ \exists \delta$ such that $\mu(E) < \delta \implies |\nu(E)| < \varepsilon$.*

As an application, take $f \in L^1$, μ any positive measure. Define:

$$\nu(E) := \int_E f\,d\mu$$

Then $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\mu(E) < \delta \implies |\nu(E)| < \varepsilon$. This is because $\nu << \mu$ as:

$$\mu(E) = 0 \implies \nu(E) = \int_E f\,d\mu = 0$$

(Notation: whenever $\nu(E) = \int_E f\,d\mu$, we write $d\nu = f\,d\mu$)

$$F(x) = \int_a^x f(y)\,dy$$

"Claim" $F'(x) = f$ for "nice" f:

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(y)\,dy$$

Let $\nu(E) = \int_E f\,dx$, then $\int_a^x f(y)\,dy = \nu([a,x])$

$$\frac{1}{h}\int_{[x,x+h]} f(y)\,dy = \frac{1}{h}\nu([x,x+h]) = \frac{\nu([x,x+h])}{\mu([x,x+h])}$$

Theorem 21 (Lebesgue-Radon-Nikodym)**.** *$(X, \mathcal{M})$ non-empty measurable space. Let ν be a σ-finite signed measure and μ be a σ-finite positive measure. Then $\exists! \lambda, \varphi$ σ-finite signed measures such that:*

$$\lambda \perp \mu,\ \varphi << \mu,\ v = \lambda + \varphi$$

Moreover, there is an integrable function $f : X \to \mathbb{R}$ such that $d\varphi = f\,d\mu$ (i.e. $\varphi(E) = \int_E f\,d\mu$). Any two such functions are equal a.e. w.r.t. μ.

i.e. for ν, μ given, unique way to write:

$$\begin{aligned}\nu(E) &= \lambda(E) + \varphi(E) \\ &= \lambda(E) + \int_E f \, d\mu\end{aligned}$$

with $\lambda \perp \mu$ and $\varphi << \mu$.

In general, given ν, μ it is not possible to write:

$$\nu(E) = \int_E f \, d\mu$$

for some f. When we can't do this we can't compute is larger $\frac{d\nu}{d\mu}$.

Proposition 31. *ν σ-finite signed μ, λ σ-finite positive. Assume $\nu << \lambda$ and $\mu << \lambda$. Then:*

$$\int h \, d\nu = \int h \frac{d\nu}{d\mu}$$

Also:

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}$$

5 Crash Course on L^p Spaces

Let $(X, \mathcal{M}, \mu)$.

$$\mathcal{L}^p = \left\{ f : X \to \mathbb{C} : \int |f|^p \, d\mu < \infty \right\} \qquad (1 \le p < \infty)$$

If $p = 1$ then $\|f\| := \int |f| \, d\mu$ is a norm on $\mathcal{L}^1$. If $p > 1$ then natural idea for norm is:

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}}$$

Again, doesn't work yet.

Define ~ (equivalence class) such that $f \sim g$ iff $f = g$ a.e. Define

$$L^p = \mathcal{L}^p / \sim \qquad (1 \le p < \infty)$$

To show $\|f\|_p$ is a norm on L^p need:

i) $\|f\|_p \ge 0$ and $\|f\|_p \iff f \equiv 0$ (in L^p).

ii) $\|\lambda f\|_p = |\lambda| \|f\|_p$ (trivial).

iii) $\|f + g\| \le \|f\|_p + \|g\|_p$

Proposition 32.

$$f \in L^p, \; g \in L^p \implies f + g \in L^p$$

Proof.

$$|f(x) + g(x)|^p \le (2 \max \{|f(x)|, |g(x)|\})^p \le 2^p (|f(x)|^p + |g(x)|^p)$$

Therefore:

$$\int |f + g|^p \, d\mu \le 2^p \int |f|^p \, d\mu + 2^p \int |g|^p \, d\mu < \infty$$

□

Lemma 4. *Let $a \geq 0$, $b \geq 0$, $0 < \lambda < 1$. Then:*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$$

With equality iff $a = b$.

Proof. This is trivial if $b = 0$, so assume $b \neq 0$, then:

$$a^\lambda b^{-\lambda} \leq \lambda \left(\frac{a}{b}\right) + 1 - \lambda$$

So want, for $t \geq 0$, $t^\lambda \leq \lambda t + 1 - \lambda$. Let $f(t) \leq 1 - \lambda$. Calculating max using differentiation gives $t = 1$ as max □

Theorem 22 (Hölder's inequality)**.**

$$\int |f \cdot g| \, d\mu \leq \|f\|_p \|g\|_q \qquad \text{(provided } \tfrac{1}{p} + \tfrac{1}{q} = 1\text{)}$$

Proof. Trivial if $\|f\|_p = 0$ or ∞, or $\|g\|_q = 0$ or ∞, so assume they are not. This it is equivalent to show:

$$\int \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_q} \, d\mu \leq 1$$

So enough to show:

$$\int |f \cdot g| \, d\mu \leq 1 \qquad \text{(whenever } \|f\|_p = \|g\|_q = 1\text{)}$$

By using the above lemma with $a = |f(x)|^p$, $b = |g(x)|^q$, $\lambda = \frac{1}{p}$, we get:

$$|f| \cdot |g| \leq \frac{1}{p}|f|^p + \frac{1}{q}|g|^q$$

Therefore:

$$\int |f \cdot g| \, d\mu \leq \int \frac{1}{p}|f|^p \, d\mu + \int \frac{1}{q}|g|^q \, d\mu = \frac{1}{p} + \frac{1}{q}$$

□

Theorem 23 (Minkowski's inequality)**.**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. (Using Hölder's inequality)
This is trivial for $p = 1$ (just triangle inequality for real numbers), so let $p > 1$:

$$\int |f+g|^p \, d\mu = \int |f+g||f+g|^{p-1} \, d\mu \leq \int (|f| + |g|)|f+g|^{p-1} \, d\mu \leq \int |f||f+g|^{p-1} \, d\mu + \int |g||f+g|^{p-1} \, d\mu$$

Now $\frac{1}{p} + \frac{1}{q} = 1 \implies \frac{1}{q} = \frac{p-1}{p}$, so:

$$\int |f+g|^p \, d\mu \underbrace{\leq}_{\text{Hölder's}} \|f\|_p \left(\int (|f+g|^{p-1})^q \, d\mu\right)^{\frac{1}{q}} + \|g\|_p \left(\int (|f+g|^{p-1})^q \, d\mu\right)^{\frac{1}{q}}$$

$$= (\|f\|_p + \|g\|_p)\left(\int |f+g|^p \, d\mu\right)^{\frac{1}{q}}$$

This gives us:

$$\left(\int |f+g|^p \, d\mu\right)^{1-\frac{1}{q}} \leq \|f\|_p + \|g\|_p$$

i.e.

$$\left(\int |f+g|^p \, d\mu\right)^{\frac{1}{p}} \leq \|f\|_p + \|g\|_p$$

□